

SPACES ADMISSIBLE FOR THE STURM-LIOUVILLE EQUATION

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ABSTRACT. We consider the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R} \quad (1)$$

where $f \in L_p^{\text{loc}}(\mathbb{R})$, $p \in [1, \infty)$ and $0 \leq q \in L_1^{\text{loc}}(\mathbb{R})$. By a solution of (1) we mean any function y , absolutely continuous together with its derivative and satisfying (1) almost everywhere in \mathbb{R} . Let positive and continuous functions $\mu(x)$ and $\theta(x)$ for $x \in \mathbb{R}$ be given. Let us introduce the spaces

$$L_p(\mathbb{R}, \mu) = \left\{ f \in L_p^{\text{loc}}(\mathbb{R}) : \|f\|_{L_p(\mathbb{R}, \mu)}^p = \int_{-\infty}^{\infty} |\mu(x)f(x)|^p dx < \infty \right\},$$

$$L_p(\mathbb{R}, \theta) = \left\{ f \in L_p^{\text{loc}}(\mathbb{R}) : \|f\|_{L_p(\mathbb{R}, \theta)}^p = \int_{-\infty}^{\infty} |\theta(x)f(x)|^p dx < \infty \right\}.$$

In the present paper, we obtain requirements to the functions μ, θ and q under which

- 1) for every function $f \in L_p(\mathbb{R}, \theta)$ there exists a unique solution (1) $y \in L_p(\mathbb{R}, \mu)$ of (1);
- 2) there is an absolute constant $c(p) \in (0, \infty)$ such that regardless of the choice of a function $f \in L_p(\mathbb{R}, \theta)$ the solution of (1) satisfies the inequality

$$\|y\|_{L_p(\mathbb{R}, \mu)} \leq c(p)\|f\|_{L_p(\mathbb{R}, \theta)}.$$

1. INTRODUCTION

In the present paper, we consider the equation

$$-y''(x) + q(x)y(x) = f(x), \quad x \in \mathbb{R} \quad (1.1)$$

where $f \in L_p^{\text{loc}}(\mathbb{R})$, $p \in [1, \infty)$ and

$$0 \leq q \in L_1^{\text{loc}}(\mathbb{R}). \quad (1.2)$$

Our general goal is to determine a space frame within which equation (1.1) always has a unique stable solution. To state the problem in a more precise way, let us fix two positive continuous functions $\mu(x)$ and $\theta(x)$, $x \in \mathbb{R}$, a number $p \in [1, \infty)$, and introduce the spaces $L_p(\mathbb{R}, \mu)$ and $L_p(\mathbb{R}, \theta)$:

$$L_p(\mathbb{R}, \mu) = \left\{ f \in L_p^{\text{loc}}(\mathbb{R}) : \|f\|_{L_p(\mathbb{R}, \mu)}^p = \int_{-\infty}^{\infty} |\mu(x)f(x)|^p dx < \infty \right\} \quad (1.3)$$

$$L_p(\mathbb{R}, \theta) = \left\{ f \in L_p^{\text{loc}}(\mathbb{R}) : \|f\|_{L_p(\mathbb{R}, \theta)}^p = \int_{-\infty}^{\infty} |\theta(x)f(x)|^p dx < \infty \right\}. \quad (1.4)$$

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For brevity, below we write $L_{p,\mu}$ and $L_{p,\theta}$, $\|\cdot\|_{p,\mu}$ and $\|\cdot\|_{p,\theta}$, instead of $L_p(\mathbb{R}, \mu)$, $L_p(\mathbb{R}, \theta)$ and $\|\cdot\|_{L_p(\mathbb{R}, \mu)}$, $\|\cdot\|_{L_p(\mathbb{R}, \theta)}$, respectively (for $\mu = 1$ we use the standard notation L_p ($L_p := L_p(\mathbb{R})$) and $\|\cdot\|_p$ ($\|\cdot\|_p := \|\cdot\|_{L_p}$)). In addition, below by a solution of (1.1) we understand any function y , absolutely continuous together with its derivative and satisfying equality (1.1) almost everywhere on \mathbb{R} .

Let us introduce the following main definition (see [12, Ch.5, §50-51]:

Definition 1.1. *We say that the spaces $L_{p,\mu}$ and $L_{p,\theta}$ make a pair $\{L_{p,\mu}, L_{p,\theta}\}$ admissible for equation (1.1) if the following requirements hold:*

- I) for every function $f \in L_{p,\theta}$ there exists a unique solution $y \in L_{p,\mu}$ of (1.1);*
- II) there is a constant $c(p) \in (0, \infty)$ such that regardless of the choice of a function $f \in L_{p,\theta}$ the solution $y \in L_{p,\mu}$ of (1.1) satisfies the inequality*

$$\|y\|_{p,\mu} \leq c(p)\|f\|_{p,\theta}. \quad (1.5)$$

Let us in addition we make the following conventions: For brevity we say “problem I)–II)” or “question on I)–II)” instead of “problem (or question) on conditions for the functions μ and θ under which requirements I)–II) of Definition 1.1 hold.” We say “the pair $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for (1.1)” instead of “the pair of spaces $\{L_{p,\mu}; L_{p,\theta}\}$ admissible for equation (1.1)”, and we often omit the word “equation” before (1.1). By c , $c(\cdot)$ we denote absolute positive constants which are not essential for exposition and may differ even within a single chain of calculations. Our general requirement (1.2) is assumed to be satisfied throughout the paper, is not referred to, and does not appear in the statements.

Let us return to Definition 1.1. The question on the admissibility of the pair $\{L_p, L_p\}$ for (1.1) was studied in [3, 6] (in [3, 6] for $\mu \equiv \theta \equiv 1$ in the case where I)–II) were valid, we said that equation (1.1) is correctly solvable in L_p . We maintain this terminology in the present paper.) Let us quote the main result of [3, 6] (in terms of Definition 1.1).

Theorem 1.2. [3] *The pair $\{L_p, L_p\}$ is admissible for (1.1) if and only if there is $a \in (0, \infty)$ such that $q_0(a) > 0$. Here*

$$q_0(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t) dt. \quad (1.6)$$

Below we continue the investigation started in [3, 6].

Our goal is as follows: given equation (1.1), to determine requirements to the weights μ and θ under which the pair $\{L_{p,\mu}; L_{p,\theta}\}$, $p \in [1, \infty)$, is admissible for (1.1). Such an approach

to the inversion of (1.1) allows to study this equation also in the case where Theorem 1.2 is not applicable, for example, in the following three cases:

- 1) $q_0(a) > 0$ for some $a \in (0, \infty)$, $f \notin L_p$, $p \in [1, \infty)$;
- 2) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \in L_p$, $p \in [1, \infty)$;
- 2) $q_0(a) = 0$ for all $a \in (0, \infty)$, $f \notin L_p$, $p \in [1, \infty)$.

Our main result (see Theorem 4.3 in §4 below) reduces the stated problem to the question on the boundedness of a certain integral operator $S : L_p \rightarrow L_p$ (see (4.3) in §4). From this criterion, under additional requirements to the functions μ , θ and q , one can deduce some concrete particular conditions which control the solution of our problem. See §4 for such restrictions.

We now describe the structure of the paper. Section 2 contains preliminaries; in Section 3 we give various technical assertions; all our results and relevant comments are presented in Section 4; all the proofs are collected in Section 5; and Section 6 contains an example of the presented statements.

2. PRELIMINARIES

Recall that our standing assumption (1.2) is not included in the statements.

Lemma 2.1. [4] *Suppose that the following condition holds:*

$$\int_{-\infty}^x q(t)dt > 0, \quad \int_x^{\infty} q(t)dt > 0, \quad \forall x \in \mathbb{R}. \quad (2.1)$$

Then for any given $x \in \mathbb{R}$, each of the equations in $d \geq 0$

$$\int_0^{\sqrt{2}d} \int_{x-t}^{x+t} q(\xi)d\xi dt = 2, \quad d \int_{x-d}^{x+d} q(\xi)d\xi = 2 \quad (2.2)$$

has a unique finite positive solution. Denote these solutions by $d(x)$ and $\hat{d}(x)$, respectively.

We have the inequalities

$$\frac{d(x)}{\sqrt{2}} \leq \hat{d}(x) \leq \sqrt{2}d(x), \quad x \in \mathbb{R}. \quad (2.3)$$

Note that the functions $d(x)$ and $\hat{d}(x)$ were introduced by the authors (see [1, 4]) and M. Otelbaev (see [14]), respectively. Analysing our assertions and requirements (see §4 below), it is useful to take into account that the function $q^*(x) \stackrel{\text{def}}{=} d^{-2}(x)$ ($d^{-2} := 1/d^2$) can be interpreted as a composed (in the sense of function theory) average of the function $q(\xi)$, $\xi \in \mathbb{R}$, at the point $\xi = x$ with step $d(x)$. Indeed, denote

$$S_x(q)(t) = \frac{1}{2t} \int_{x-t}^{x+t} q(\xi)d\xi, \quad t > 0, \quad x \in \mathbb{R},$$

$$M(f)(\eta) = \frac{1}{\eta^2} \int_0^{\sqrt{2}\eta} t f(t) dt, \quad \eta > 0.$$

Clearly, $S_x(q)(t)$ is the Steklov average with step $t > 0$ of the function $q(\xi)$, $\xi \in \mathbb{R}$, at the point $\xi = x$, and $M(f)(\eta)$ is the average of the function $f(t)$, $t > 0$ with step $\eta > 0$ at the point $t = 0$. Now, using

$$\begin{aligned} q^*(x) &= \frac{1}{d^2(x)} = \frac{1}{2d^2(x)} \int_0^{\sqrt{2}d(x)} \int_{x-t}^{x+t} q(\xi) d\xi dt \\ &= \frac{1}{d^2(x)} \int_0^{\sqrt{2}d(x)} t \left[\frac{1}{2t} \int_{x-t}^{x+t} q(\xi) d\xi \right] dt = M(S_x(q))(d(x)). \end{aligned}$$

Similarly, the function $\hat{q}^*(x) \stackrel{\text{def}}{=} \hat{d}(x)^{-2} x \in \mathbb{R}$, can be interpreted as the Steklov average of the function $q(\xi)$, $\xi \in \mathbb{R}$, at the point $\xi = x$ with step $\hat{d}(x)$. Indeed (see (2.1)), we have

$$\hat{q}^*(x) = \frac{1}{\hat{d}^2(x)} = \frac{1}{2\hat{d}(x)} \int_{x-\hat{d}(x)}^{x+\hat{d}(x)} q(\xi) d\xi = S_x(q)(\hat{d}(x)).$$

Theorem 2.2. [2] *Suppose that (2.1) holds. Then the equation*

$$z''(x) = q(x)z(x), \quad x \in \mathbb{R}, \quad (2.4)$$

has a fundamental system of solutions (FSS) $\{u(x), v(x)\}$, $x \in \mathbb{R}$, such that

$$u(x) > 0, \quad v(x) > 0, \quad u'(x) < 0, \quad v'(x) > 0, \quad \forall x \in \mathbb{R}, \quad (2.5)$$

$$v'(x)u(x) - u'(x)v(x) = 1, \quad \forall x \in \mathbb{R}, \quad (2.6)$$

$$\lim_{x \rightarrow -\infty} \frac{v(x)}{u(x)} = \lim_{x \rightarrow \infty} \frac{u(x)}{v(x)} = 0, \quad (2.7)$$

$$|\rho'(x)| < 1, \quad \forall x \in \mathbb{R}, \quad \rho(x) \stackrel{\text{def}}{=} u(x)v(x). \quad (2.8)$$

Let us introduce the Green function of equation (1.1):

$$G(x, t) = \begin{cases} u(x)v(t), & x \geq t \\ u(t)v(x), & x \leq t \end{cases} \quad (2.9)$$

Theorem 2.3. [8] *For $x, t \in \mathbb{R}$, we have the Davies-Harrell representations for the solution $\{u(x), v(x)\}$ and the Green function $G(x, t)$:*

$$u(x) = \sqrt{\rho(x)} \exp\left(-\frac{1}{2} \int_{x_0}^x \frac{d\xi}{\rho(\xi)}\right), \quad v(x) = \sqrt{\rho(x)} \exp\left(\frac{1}{2} \int_{x_0}^x \frac{d\xi}{\rho(\xi)}\right), \quad (2.10)$$

$$G(x, t) = \sqrt{\rho(x)\rho(t)} \exp\left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right). \quad (2.11)$$

Here x_0 is a unique solution of the equation $u(x) = v(x)$, $x \in \mathbb{R}$ (see [2]), the function ρ is defined in (2.8).

Theorem 2.4. [4] *Suppose that (2.1) holds. Then we have the Otelbaev inequalities:*

$$\frac{d(x)}{2\sqrt{2}} \leq \rho(x) \leq \sqrt{2}d(x), \quad x \in \mathbb{R}. \quad (2.12)$$

Two-sided, sharp by order estimates of the function ρ were first obtained by M. Otelbaev (see [14]), and therefore all such inequalities are referred to by his name. Note that the inequalities given in [14] are expressed in terms of another auxiliary function, more complicated than $d(x)$, $x \in \mathbb{R}$, and are proven under auxiliary requirements to the function q .

Let us introduce the Green operator

$$(Gf)(x) = \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad x \in \mathbb{R}. \quad (2.13)$$

Theorem 2.5. [6] *Suppose that (2.1) holds, and let $p \in [1, \infty)$. Then equation (1.1) is correctly solvable in L_p (or, in other words, the pair $\{L_p, L_p\}$ is admissible for (1.1)) if and only if the operator $G : L_p \rightarrow L_p$ is bounded. In the latter case, for $f \in L_p$, the solution $y \in L_p$ of (1.1) is of the form $y = Gf$.*

Theorem 2.6. [3] *For $p \in [1, \infty)$, equation (1.1) is correctly solvable in L_p (i.e., the pair $\{L_p, L_p\}$ is admissible for (1.1)) if and only if equalities (2.1) hold and $\hat{d}_0 < \infty$. Here*

$$\hat{d}_0 = \sup_{x \in \mathbb{R}} \hat{d}(x). \quad (2.14)$$

Theorem 2.7. [11] *Let μ and θ be continuous positive functions in \mathbb{R} , and let H be an integral operator*

$$(Hf)(t) = \mu(t) \int_t^{\infty} \theta(\xi)f(\xi)d\xi, \quad t \in \mathbb{R}. \quad (2.15)$$

For $p \in (1, \infty)$, the operator $H : L_p \rightarrow L_p$ is bounded if and only if $H_p < \infty$. Here $H_p = \sup_{x \in \mathbb{R}} H_p(x)$,

$$H_p(x) = \left(\int_{-\infty}^x \mu(t)^p dt \right)^{1/p} \cdot \left(\int_x^{\infty} \theta(t)^{p'} dt \right)^{1/p'}, \quad p' = \frac{p}{p-1}. \quad (2.16)$$

In addition,

$$H_p \leq \|H\|_{p \rightarrow p} \leq (p)^{1/p} (p')^{1/p'} H_p. \quad (2.17)$$

Theorem 2.8. [11] *Let μ and θ be continuous positive functions in \mathbb{R} , and let \tilde{H} be an integral operator*

$$(\tilde{H}f)(t) = \mu(t) \int_{-\infty}^t \theta(\xi)f(\xi)d\xi, \quad t \in \mathbb{R}. \quad (2.18)$$

For $p \in (1, \infty)$ the operator $\tilde{H} : L_p \rightarrow L_p$ is bounded if and only if $\tilde{H}_p < \infty$. Here $\tilde{H}_p = \sup_{x \in \mathbb{R}} \tilde{H}_p(x)$

$$\tilde{H}_p(x) = \left[\int_{-\infty}^x \theta(t)^{p'} dt \right]^{1/p'} \cdot \left[\int_x^{\infty} \mu(t)^p dt \right]^{1/p}, \quad p' = \frac{p}{p-1}. \quad (2.19)$$

In addition,

$$\tilde{H}_p \leq \|\tilde{H}\|_{p \rightarrow p} \leq (p)^{1/p} (p')^{1/p'} \tilde{H}_p. \quad (2.20)$$

Theorem 2.9. [10] Let $-\infty \leq a < b \leq \infty$, let $K(x, t)$ be a continuous function for $s, t \in (a, b)$, and let K be an integral operator

$$(Kf)(t) = \int_a^b K(s, t) f(s) ds, \quad t \in (a, b). \quad (2.21)$$

Then we have the inequality

$$\|K\|_{L_1(a,b) \rightarrow L_1(a,b)} = \sup_{s \in (a,b)} \int_a^b |K(s, t)| dt. \quad (2.22)$$

3. AUXILIARY ASSERTIONS

In this section, we mainly present the properties of the function $d(x)$, $x \in \mathbb{R}$ (see Lemma 2.1). Here we assume that condition (2.1) is satisfied, and we do not include it in the statements.

Lemma 3.1. The function $d(x)$ is continuously differentiable for all $x \in \mathbb{R}$, and the following inequality holds:

$$\sqrt{2}|d'(x)| \leq 1, \quad x \in \mathbb{R}. \quad (3.1)$$

Remark 3.2. It is interesting to compare estimate (2.8) (see also (2.12)) with estimate (3.1).

Lemma 3.3. For $x \in \mathbb{R}$, we have the inequalities

$$4^{-1}d(x) \leq d(t) \leq 4d(x), \quad \text{if } |t - x| \leq d(x). \quad (3.2)$$

Lemma 3.4. For $x \in \mathbb{R}$, we have the inequalities (see Theorem 2.2):

$$c^{-1} \leq \frac{u(t)}{u(x)}; \quad \frac{v(t)}{v(x)}; \quad \frac{\rho(t)}{\rho(x)} \leq c \quad \text{if } |t - x| \leq d(x). \quad (3.3)$$

Lemma 3.5. For a given $x \in \mathbb{R}$, consider the function

$$F(\eta) = \int_0^{\sqrt{2}\eta} \int_{x-t}^{x+t} q(\xi) d\xi dt, \quad \eta \geq 0. \quad (3.4)$$

The function $F(\eta)$ is differentiable and non-negative, together with its derivative, and

$$F(0) = 0, \quad F(\infty) = \infty. \quad (3.5)$$

In addition, the inequality $\eta \geq d(x)$ ($0 \leq \eta \leq d(x)$) holds if and only if $F(\eta) \geq 2$ ($F(\eta) \leq 2$).

Lemma 3.6. *Let a function f be defined on \mathbb{R} and absolutely continuous together with its derivative. Then for all $x \in \mathbb{R}$ and $t \geq 0$, we have the equality*

$$\int_{x-t}^{x+t} f(\xi) d\xi = 2f(x)t + \int_0^t \int_0^{t_1} \int_{x-t_2}^{x+t_2} f''(t_3) dt_3 dt_2 dt_1. \quad (3.6)$$

Theorem 3.7. *Suppose that condition (2.1) holds and the function $q(x)$ can be written in the form*

$$q(x) = q_1(x) + q_2(x), \quad x \in \mathbb{R}, \quad (3.7)$$

where $q_1(x)$, $x \in \mathbb{R}$, is positive and absolutely continuous together with its derivative, and $q_2 \in L_1^{\text{loc}}(\mathbb{R})$. Denote

$$A(x) = \left[0, \frac{2}{\sqrt{q_1(x)}} \right], \quad x \in \mathbb{R}, \quad (3.8)$$

$$\varkappa_1(x) = \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi) d\xi \right|, \quad x \in \mathbb{R}, \quad (3.9)$$

$$\varkappa_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right|, \quad x \in \mathbb{R}. \quad (3.10)$$

If we have the condition

$$\varkappa_1(x) \rightarrow 0, \quad \varkappa_2(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty, \quad (3.11)$$

then the following relations hold:

$$d(x)\sqrt{q_1(x)} = 1 + \varepsilon(x), \quad |\varepsilon(x)| \leq 2(\varkappa_1(x) + \varkappa_2(x)), \quad |x| \gg 1, \quad (3.12)$$

$$c^{-1} \leq d(x)\sqrt{q_1(x)} \leq c \quad \text{for all} \quad x \in \mathbb{R}. \quad (3.13)$$

4. MAIN RESULTS

Throughout the sequel we assume that our standing requirements to the functions q (see (1.2)), and μ and θ (see §1) are satisfied, and we do not mention them in the statements.

Theorem 4.1. *Suppose that the function q is nonnegative and continuous at every point of the real axis. Suppose that for a given $p \in [1, \infty)$ the following condition holds:*

$$\int_{-\infty}^0 \mu(t)^p dt = \int_0^{\infty} \mu(t)^p dt = \infty. \quad (4.1)$$

Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1) only if inequalities (2.1) hold.

To make our a priori requirements independent of the parameter $p \in [1, \infty)$, throughout the sequel we assume that together with (1.2), condition (2.1) holds. Similar to (1.2), below this condition is not quoted and does not appear in the statements.

Lemma 4.2. *Suppose that the following condition holds:*

$$\int_{-\infty}^0 \mu(t)dt = \int_0^{\infty} \mu(t)dt = \infty. \quad (4.2)$$

Then for every $p \in [1, \infty)$ equation (2.4) has no solutions $z \in L_{p,\mu}$ apart from $z \equiv 0$.

Note that for $\mu \equiv 1$ Lemma 4.2 was proved in [2].

Our main result is the following.

Theorem 4.3. *Suppose that condition (4.2) holds. Then the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1) if and only if the operator $S : L_p \rightarrow L_p$ is bounded. Here*

$$(Sf)(x) = \mu(x) \int_{-\infty}^{\infty} \frac{G(x,t)}{\theta(t)} f(t)dt, \quad x \in \mathbb{R}, \quad f \in L_p. \quad (4.3)$$

Note that for $\mu \equiv \theta \equiv 1$ Theorem 4.3 was proved in [6]. Thus, this theorem reduces the original problem on the admissibility of the pair $\{L_{p,\mu}; L_{p,\theta}\}$ for (1.1) to the boundedness of the integral operator $S : L_p \rightarrow L_p$ (see (4.3)). This result is clearly useful for the investigation of (1.1) for the following reason. Consider, say, the case $p \in (1, \infty)$. The operator S is a sum of two operators of Hardy type (see (2.9), (2.15) and (2.18)):

$$(S_1 f)(x) = \mu(x) u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} f(t)dt, \quad x \in \mathbb{R}, \quad (4.4)$$

$$(S_2 f)(x) = \mu(x) v(x) \int_x^{\infty} \frac{u(t)}{\theta(t)} f(t)dt, \quad x \in \mathbb{R}. \quad (4.5)$$

For the norms $\|S_1\|_{p \rightarrow p}$, $\|S_2\|_{p \rightarrow p}$, we know sharp by order two-sided estimates (see (2.17) and (2.20)), which can be expressed in terms of the weights μ, θ and a FSS $\{u, v\}$ of equation (2.4). The solutions $\{u, v\}$ can, in turn, be expressed in terms of the implicit function ρ (see (2.10)), for which in turn one has sharp by order estimates in terms of the function d (see (2.12) and (2.2)). Finally, for the implicit function d , which is, in general, not computable, as well as the function ρ , we have sharp by order two-sided estimates, which can be expressed in terms of the original function q (see (3.12), (3.13)). Thus, this long chain of estimates yields some information allowing us to find conditions for the boundedness of the operator $S_i : L_p \rightarrow L_p$, $i = 1, 2$ (and hence of the operator $S : L_p \rightarrow L_p$, $p \in (1, \infty)$), which are expressed in terms of the weights μ, θ and the function q . We want to emphasize that these conditions become precise if we are able to use the information obtained from the estimates in an ingenious way (see, say, [6] where similar arguments were used). One can compare this approach to that of applying the Cauchy criterion for the convergence of a number series to getting various working criteria, convenient for practical investigation of a given number

series. In a similar way, Theorem 4.3 can be used for deducing convenient particular tests for the admissibility of the pair $\{L_{p,\mu}; L_{p,\theta}\}$, $p \in [1, \infty)$, for a given equation (1.1).

Here is an example. The assertion given below (Theorem 4.7) is obtained by using one of the possible ways for practical implementation of the approach to the study of (1.1) presented above.

To formulate Theorem 4.7, we need some new definitions, auxiliary assertions and comments.

Definition 4.4. *We say that the function q belongs to the class H (and write $q \in H$) if the following equality holds:*

$$\lim_{|x| \rightarrow \infty} \nu(x) = 0. \quad (4.6)$$

Here

$$\nu(x) = d(x) \int_0^{\sqrt{2}d(x)} (q(x+t) - q(x-t))dt, \quad x \in \mathbb{R}. \quad (4.7)$$

In the next assertion, we state an important property of the functions $q \in H$.

Lemma 4.5. *Let $q \in H$. Then for any $\varepsilon > 0$ there is a constant $c(\varepsilon) \in [1, \infty)$ such that for all $x, t \in \mathbb{R}$ the following inequalities hold:*

$$c(\varepsilon)^{-1} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \leq \frac{d(t)}{d(x)} \leq c(\varepsilon) \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \quad (4.8)$$

Note that for $\varepsilon \geq 1/\sqrt{2}$ inequalities (4.8) hold regardless of condition (4.6). Indeed, under conditions (1.2) and (2.1), the function $d(x)$, $x \in \mathbb{R}$ is well-defined, differentiable, and satisfies the following relations (see Lemmas 2.1 and 3.1:

$$\begin{aligned} -\varepsilon &\leq -\frac{1}{\sqrt{2}} \leq d'(\xi) \leq \frac{1}{\sqrt{2}} \leq \varepsilon, \quad \xi \in \mathbb{R} \quad \Rightarrow \\ &-\frac{\varepsilon}{d(\xi)} \leq \frac{d'(\xi)}{d(\xi)} \leq \frac{\varepsilon}{d(\xi)}, \quad \xi \in \mathbb{R} \quad \Rightarrow \\ \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) &\leq \frac{d(t)}{d(x)} \leq \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right), \quad x, t \in \mathbb{R}. \end{aligned} \quad (4.9)$$

This means that in contrast with (4.9), for $\varepsilon \in (0, 1/\sqrt{2})$ estimates (4.8) arise because of condition (4.6).

Definition 4.6. *Let $q \in H$. We say that a pair of weights (weight functions) $\{\mu, \theta\}$ agrees with the function q if for any $\varepsilon > 0$ there is a constant $c(\varepsilon) \in [1, \infty)$ such that for all $t, x \in \mathbb{R}$*

one has the inequalities

$$c(\varepsilon)^{-1} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \leq \sqrt{\frac{d(t)}{d(x)} \frac{\mu(t)}{\mu(x)}}; \quad \sqrt{\frac{d(t)}{d(x)} \frac{\theta(x)}{\theta(t)}} \leq c(\varepsilon) \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \quad (4.10)$$

In the latter case, we say that the pair $\{L_{p,\mu}; L_{p,\theta}\}$, $p \in [1, \infty)$, agrees with equation (1.1)

Theorem 4.7. Suppose that conditions (4.2) hold. Let $q \in H$. Suppose that the pair $\{L_{p,\mu}; L_{p,\theta}\}$, $p \in [1, \infty)$ agrees with equation (1.1). Then this pair is admissible for (1.1) if and only if $m(q, \mu, \theta) < \infty$. Here

$$m(q, \mu, \theta) = \sup_{x \in \mathbb{R}} \left(\frac{\mu(x)}{\theta(x)} d^2(x) \right). \quad (4.11)$$

To prove inequalities (4.10), the following lemma can be useful.

Lemma 4.8. Suppose that a function $\mu(x)$ is defined, positive and differentiable for all $x \in \mathbb{R}$, let $q \in H$, and let $d(x)$, $x \in \mathbb{R}$, denote the auxiliary function from Lemma 2.1. Then, if the equality

$$\lim_{|x| \rightarrow \infty} \frac{\mu'(x)}{\mu(x)} d(x) = 0 \quad (4.12)$$

holds, then for any given $\varepsilon > 0$ there is a constant $c(\varepsilon) \in (0, \infty)$ such that for all $t, x \in \mathbb{R}$ inequalities (4.10) hold.

The next assertions are convenient for the study of concrete equations. They are obvious and are given without proofs.

Theorem 4.9. Let $q \in H$, and suppose that

$$d_0 \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} d(x) = \infty, \quad (4.13)$$

$$\int_{-\infty}^0 q^*(x) dx = \int_0^{\infty} q^*(x) dx = \infty, \quad q^*(x) = \frac{1}{d^2(x)}, \quad x \in \mathbb{R}. \quad (4.14)$$

Then the following assertions hold:

- A) for $p \in [1, \infty)$ the pair $\{L_p; L_p\}$ is not admissible for (1.1);
- B) for $p \in [1, \infty)$ the pair $\{L_{p,q^*}; L_p\}$ is admissible for (1.1).

Theorem 4.10. Let $q \in H$, and suppose that the weight function $\theta(x)$, $x \in \mathbb{R}$, is such that $m_0 > 0$ where

$$m_0 = \inf_{x \in \mathbb{R}} (q^*(x) \theta(x)), \quad q^*(x) = \frac{1}{d^2(x)}. \quad (4.15)$$

Then for $p \in [1, \infty)$ the pair $\{dL_p; L_{p,\theta}\}$ is admissible for (1.1).

5. PROOFS

Proof of Lemma 3.1. The existence of the derivative $d'(x)$, $x \in \mathbb{R}$ is a consequence of the theory of implicit functions [7, Ch.II,§1,no.3]. It is proven in the same way as in [5]. The following relations are deduced from (2.2):

$$\begin{aligned} & \int_0^{\sqrt{2}d(x)} \int_{x-t}^{x+t} q(\xi) d\xi dt = 2 \quad \Rightarrow \\ 0 &= \sqrt{2}d'(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi + \int_0^{\sqrt{2}d(x)} [q(x+t) - q(x-t)] dt \\ &= \sqrt{2}d'(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi + \left[\int_x^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^x q(\xi) d\xi \right] \Rightarrow \\ |d'(x)| &= \frac{1}{\sqrt{2}} \left| \int_x^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^x q(\xi) d\xi \right| \left(\int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi \right)^{-1} \leq \frac{1}{\sqrt{2}}. \end{aligned}$$

□

Proof of Lemma 3.3. Below we use Lagrange's formula and (3.1):

$$\begin{aligned} |d(t) - d(x)| &= |d'(\theta)| |t - x| \leq \frac{d(x)}{\sqrt{2}} \quad \Rightarrow \\ d(t) &\leq \left(1 + \frac{1}{\sqrt{2}}\right) d(x) \leq 4d(x) \quad \text{for } t \in [x - d(x), x + d(x)] \\ d(t) &\geq \left(1 - \frac{1}{\sqrt{2}}\right) d(x) \geq \frac{d(x)}{4} \quad \text{for } t \in [x - d(x), x + d(x)]. \end{aligned}$$

□

Proof of Lemma 3.4. Below we use (2.12) and (3.2):

$$\begin{aligned} \int_{x-d(x)}^{x+d(x)} \frac{d(\xi)}{\rho(\xi)} &= \int_{x-d(x)}^{x+d(x)} \frac{d(\xi)}{\rho(\xi)} \cdot \frac{d(x)}{d(\xi)} \cdot \frac{d\xi}{d(x)} \leq 2\sqrt{2} \cdot 4 \cdot 2 = c < \infty, \\ \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{\rho(\xi)} &= \int_{x-d(x)}^{x+d(x)} \frac{d(\xi)}{\rho(\xi)} \cdot \frac{d(x)}{d(\xi)} \cdot \frac{d\xi}{d(x)} \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{4} \cdot 2 \geq c^{-1} > 0. \end{aligned}$$

Now we use this together with (2.10) and obtain

$$\begin{aligned} \frac{u(t)}{u(x)} &\geq \sqrt{\frac{\rho(t)}{\rho(x)}} \exp\left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) \geq \sqrt{\frac{d(x)}{\rho(x)} \cdot \frac{d(t)}{d(x)} \cdot \frac{\rho(t)}{d(t)}} \exp\left(-\frac{1}{2} \left| \int_{x-d}^{x+d} \frac{d\xi}{\rho(\xi)} \right| \right) \\ &\geq c^{-1} > 0; \\ \frac{u(t)}{u(x)} &\leq \sqrt{\frac{\rho(t)}{\rho(x)}} \exp\left(\frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) \leq \sqrt{\frac{d(x)}{\rho(x)} \cdot \frac{d(t)}{d(x)} \cdot \frac{\rho(t)}{d(t)}} \exp\left(\frac{1}{2} \left| \int_{x-d}^{x+d} \frac{d\xi}{\rho(\xi)} \right| \right) \\ &\geq c < \infty. \end{aligned}$$

Inequalities (3.3) for the solution v are checked similarly, and estimates (3.3) for ρ follow from the estimates of u and v and (2.8). \square

Proof of Lemma 3.5. To prove that the function $F(\eta)$ is differentiable and the functions $F(\eta)$ and $F'(\eta)$ are non-negative for $\eta \geq 0$, we use properties of integral. The last assertion of the lemma follows from Lagrange's formula and the relations

$$F(\eta) - 2 = F(\eta) - F(d(x)) = F'(\theta)(\eta - d(x)).$$

\square

Proof of Lemma 3.6. To obtain (3.6), we use the following simple transformations

$$\begin{aligned} \int_{x-t}^{x+t} f(\xi) d\xi &= \int_0^t [f(x+t_1) + f(x-t_1)] dt_1 = 2f(x)t + \int_0^t [f(x+t_1) - f(x)] dt \\ &\quad - \int_0^t [f(x) - f(x-t_1)] dt_1 = 2f(x)t + \int_0^t \left[\int_0^{t_1} (f(x+t_2))' dt_2 \right] dt_1 \\ &\quad - \int_0^t \left[\int_0^{t_1} (f(x-t_2))' dt_2 \right] dt_1 = 2f(x)t + \int_0^t \int_0^{t_1} [f(x+t_2) - f(x-t_2)]' dt_2 dt_1 \\ &= 2f(x)t + \int_0^t \int_0^{t_1} \int_{x-t_2}^{x+t_2} f''(t_3) dt_3 dt_2 dt_1. \end{aligned}$$

\square

Proof of Theorem 3.7. Set

$$\eta(x) = \frac{1 - \delta(x)}{\sqrt{q_1(x)}}, \quad \delta(x) = 2(\varkappa_1(x) + \varkappa_2(x)), \quad |x| \gg 1.$$

Then by (3.4), (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), we have

$$\begin{aligned} F(\eta(x)) &= \int_0^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_1(\xi) d\xi dt + \int_0^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_2(\xi) d\xi dt \\ &\leq \int_0^{\sqrt{2}\eta(x)} \left[2q_1(x)t + \int_0^t \int_0^{t_1} \int_{x-t_2}^{x+t_2} q_1''(t_3) dt_3 dt_2 dt_1 \right] \\ &\quad + \sqrt{2}\eta(x) \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right| \leq \left(\sqrt{2}\eta(x) \right)^2 q_1(x) \\ &\quad + \frac{(\sqrt{2}\eta(x))^3}{6} \sup_{t_2 \in A(x)} \left| \int_{x-t_2}^{x+t_2} q_1''(\xi) d\xi \right| + \sqrt{2}(1 - \delta(x))\varkappa_2(x) \\ &\leq 2(1 - \delta(x))^2 + \frac{\sqrt{2}}{3}(1 - \delta(x))^3 \varkappa_1(x) + \sqrt{2}\varkappa_2(x) \\ &\leq 2[(1 - \delta(x))^2 + \varkappa_1(x) + \varkappa_2(x)] \\ &= 2 \left[1 - \frac{\delta(x)}{2} - \left(\frac{\delta(x)}{2} - \delta^2(x) \right) - \varkappa_1(x) - \varkappa_2(x) \right] \leq 2. \end{aligned}$$

Hence $d(x) \geq \eta(x)$ for $|x| \gg 1$ by Lemma 3.5. Let now

$$\eta(x) = \frac{1 + \delta(x)}{\sqrt{q_1(x)}}, \quad \delta(x) = 2(\kappa_1(x) + \kappa_2(x)), \quad |x| \gg 1.$$

Then by the same arguments we obtain:

$$\begin{aligned} F(\eta(x)) &= \int_0^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_1(\xi) d\xi dt + \int_0^{\sqrt{2}\eta(x)} \int_{x-t}^{x+t} q_2(\xi) d\xi dt \\ &\geq \int_0^{\sqrt{\eta(x)}} \int_{x-t}^{x+t} \left[2q_1(x)t + \int_0^t \int_0^{t_1} \int_{x-t_2}^{x+t_2} q_1''(t_3) dt_3 dt_2 dt_1 \right] dt \\ &\quad - \sqrt{2}\eta(x) \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right| \geq \left(\sqrt{2}\eta(x) \right)^2 q_1(x) \\ &\quad - \frac{(\sqrt{2}\eta(x))^3}{6} \sup_{t_2 \in A(x)} \left| \int_{x-t_2}^{x+t_2} q_1''(t_3) dt_3 \right| - \sqrt{2}(1 + \delta(x))\kappa_2(x) \\ &\geq 2(1 + \delta(x))^2 - \frac{\sqrt{2}}{3}(1 + \delta(x))^3 \kappa_1(x) - 2\kappa_2(x) \\ &\geq 2(1 + \delta(x)) + \kappa_1(x) + \kappa_2(x) \geq 2. \end{aligned}$$

Hence $d(x) \leq \eta(x)$ for $|x| \gg 1$ by Lemma 3.5, and equality (3.12) is proven. Further, since the function $d(x)\sqrt{q_1(x)}$ is continuous and positive for all $x \in \mathbb{R}$, for all $x_0 \in (0, \infty)$ we the inequalities:

$$\begin{aligned} 0 < m \leq f(x) \leq M < \infty, \quad |x| \leq x_0 \\ m = \min_{|x| \leq x_0} f(x), \quad M = \max_{|x| \leq x_0} f(x), \quad f(x) = d(x)\sqrt{q_1(x)}. \end{aligned}$$

Together with (3.12), this implies (3.13). \square

Proof of Theorem 4.1. Assume the contrary. Then (4.1) holds, the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), and there exists $x_0 \in \mathbb{R}$ such that one of inequalities (2.1), say, the second one, does not hold:

$$\int_{x_0}^{\infty} q(t) dt = 0 \quad \Rightarrow \quad q(x) \equiv 0, \quad x \in [x_0, \infty). \quad (5.1)$$

Without loss of generality, in what follows we assume $x_0 \geq 1$. Let us introduce the functions φ and f_0 .

$$1) \quad \varphi \in C^\infty(\mathbb{R}), \quad \text{supp } \varphi = [x_0, \infty), \quad 0 \leq \varphi(x) \leq 1 \quad \text{for } x \in \mathbb{R}, \quad (5.2)$$

$$\varphi(x) \equiv 1 \quad \text{for } x \geq x_0 + 1 \quad (5.3)$$

$$2) \quad f_0(x) := -\varphi''(x) + q(x)\varphi(x), \quad x \in \mathbb{R}. \quad (5.4)$$

From 1)–2) we obtain the equality

$$\begin{aligned} q(x)\varphi(x) &\equiv 0, \quad x \in \mathbb{R} \quad \Rightarrow \\ f_0(x) &= -\varphi''(x), \quad x \in \mathbb{R} \quad \Rightarrow \quad \text{supp } f_0 = [x_0, x_0 + 1]. \end{aligned} \quad (5.5)$$

According to (5.5), we conclude that $f_0 \in L_{p,\theta}$:

$$\|f_0\|_{L_{p,\theta}}^p = \int_{-\infty}^{\infty} |\theta(x)f_0(x)|^p dx = \int_{x_0}^{x_0+1} |\theta(x)\varphi''(x)|^p dx = c(x_0) < \infty.$$

Since the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), we conclude that (1.1) for $f = f_0$ has a unique solution $y_0 \in L_{p,\mu}$. Then (see (5.4) and (5.5))

$$y_0(x) = \varphi(x) + z(x), \quad x \in \mathbb{R}, \quad (5.6)$$

where $z(x)$, $x \in \mathbb{R}$, is some solution of (2.4). From (2.4) and (5.1), we obtain the equality

$$z''(x) = 0 \quad \text{for } x \in [x_0, \infty) \quad \Rightarrow \quad z(x) = c_1 + c_2x \quad \text{for } x \geq x_0. \quad (5.7)$$

Let us show that $c_2 = 0$. Assume to the contrary that $c_2 \neq 0$. Choose x_1 so that to have the inequality

$$\frac{|1 + c_1|}{|c_2|} \cdot \frac{1}{x} \leq \frac{1}{2} \quad \text{for } x \geq x_1 \geq x_0 + 1. \quad (5.8)$$

Then (see (5.3))

$$\begin{aligned} \infty &> \|y_0\|_{p,\mu}^p \geq \int_{x_1}^{\infty} \mu(x)^p |\varphi(x) + z(x)|^p dx = \int_{x_1}^{\infty} \mu(x)^p |1 + c_1 + c_2x|^p dx \\ &\geq |c_2x_1|^p \int_{x_1}^{\infty} \mu(x)^p \left| 1 - \left| \frac{1 + c_1}{c_2} \right| \frac{1}{x} \right|^p dx \geq \left| \frac{c_2x_1}{2} \right|^p \int_{x_1}^{\infty} \mu(x)^p dx = \infty, \end{aligned}$$

and we get a contradiction. Hence $c_2 = 0$. Let us check that also $c_1 = 0$. Assume that $c_1 \neq 0$. Since $\varphi \in C^\infty(\mathbb{R})$, from (5.2) it follows that $\varphi(x_0) = \varphi'(x_0) = 0$ and therefore (see (5.7)):

$$\begin{aligned} y(x_0) &= \varphi(x_0) + z(x_0) = c_1, \\ y'(x_0) &= \varphi'(x_0) + z'(x_0) = 0. \end{aligned}$$

In addition, $\varphi(x) \equiv 0$ for $x \leq x_0$, and therefore from (5.5) and (5.6) it follows that the function z is a solution of the Cauchy problem

$$\begin{cases} z''(x) = q(x)z(x), & x \leq x_0 \end{cases} \quad (5.9)$$

$$\begin{cases} z(x_0) = c_1, \quad z'(x_0) = 0. \end{cases} \quad (5.10)$$

Further, without loss of generality, we assume that $c_1 = 1$. Let us check that then we have the inequality

$$z(x) \geq 1 \quad \text{for } x \leq x_0. \quad (5.11)$$

Towards this end, first note that since $z(x_0) = 1$, we have $z(x) > 0$ in some left half-neighborhood of the point x_0 (i.e., for $x \in (x_0 - \varepsilon, x_0]$ for some $\varepsilon > 0$). But then $z(x) > 0$ for all $x < x_0$. Indeed, if this is not the case, then $z(x)$ has at least one zero on $(-\infty, x_0)$. Let \tilde{x} be the first zero of $z(x)$ to the left from x_0 . Then $z'(\tilde{x}) \geq 0$. Indeed, if $z'(\tilde{x}) < 0$ and $z(\tilde{x}) = 0$, then $z(x) < 0$ in some right half-neighborhood of the \tilde{x} . But $z(x_0) = 1$ and $\tilde{x} < x_0$. Hence, the interval (\tilde{x}, x_0) contains a zero of $z(x)$, contrary to the definition of the point \tilde{x} . Thus $z'(\tilde{x}) \geq 0$. On the other hand,

$$\begin{aligned} z'(x_0) - z'(\tilde{x}) &= \int_{\tilde{x}}^{x_0} q(\xi)z(\xi)d\xi \quad \Rightarrow \\ z'(\tilde{x}) &= - \int_{\tilde{x}}^{x_0} q(\xi)z(\xi)d\xi \leq 0. \end{aligned}$$

Hence $z'(\tilde{x}) = 0$. But then the function $z(x)$ is a solution of the Cauchy problem

$$\begin{aligned} z''(x) &= q(x)z(x), \quad x \leq x_0 \\ z(\tilde{x}) &= z'(\tilde{x}) = 0 \\ \Rightarrow \quad z(x) &\equiv 0, \quad x \leq x_0. \end{aligned}$$

We get a contradiction because $z(x_0) = 1$. Thus $z(x) > 0$ for $x \leq x_0$. Then for $x \leq x_0$, we have

$$-z'(x) = z'(x_0) - z'(x) = \int_x^{x_0} q(\xi)z(\xi)d\xi \geq 0 \quad \Rightarrow \quad z'(x) \leq 0, \quad x \leq x_0.$$

Hence $z(x) \geq z(x_0) = 1$ for $x \leq x_0$. This implies that

$$\begin{aligned} \infty > \|y_0\|_{p,\mu} &= \int_{-\infty}^{\infty} |\mu(x)y_0(x)|^p dx \geq \int_{-\infty}^{x_0} |\mu(x)y_0(x)|^p dx \\ &= \int_{-\infty}^{x_0} |\mu(x)z(x)|^p dx \geq \int_{-\infty}^{x_0} \mu(x)^p dx = \infty. \end{aligned}$$

We get a contradiction. Hence $c_1 = 0$, and we obtain the equality

$$\begin{aligned} y_0(x) &= \varphi(x), \quad x \geq x_0 \quad \Rightarrow \\ \infty > \|y_0\|_{p,\mu}^p &\geq \int_{x_0+1}^{\infty} |\mu(x)y_0(x)|^p dx = \int_{x_0+1}^{\infty} |\mu(x)\varphi(x)|^p dx \\ &= \int_{x_0+1}^{\infty} \mu(x)^p dx = \infty. \end{aligned}$$

We get a contradiction. Hence (5.1) does not hold. \square

Proof of Lemma 4.2. Let us show that in the case of (4.2) for all $p \in [1, \infty)$ we have the equalities

$$\int_{-\infty}^0 (\mu(t)u(t))^p dt = \int_0^{\infty} (\mu(t)v(t))^p dt = \infty. \quad (5.12)$$

We only consider the second equality because the first one can be proved in the same way. For $p = 1$ equality (5.11) follows from Theorem 2.2 and (4.2) is a straightforward manner. Let $p \in (1, \infty)$, $p' = p(p-1)^{-1}$. The following relations rely only on Theorem 2.2:

$$\begin{aligned} \int_0^\infty \frac{dt}{v(t)^{p'}} &= \int_0^\infty \frac{v'(t)v(t)^{-p'}}{v'(t)} dt \leq \frac{1}{v'(0)} \int_0^\infty v'(t)v(t)^{-p'} dt \\ &= \frac{1}{p'-1} \frac{1}{v'(0)} \left(\frac{1}{v(0)^{p'-1}} - \frac{1}{v(\infty)^{p'-1}} \right) \leq \frac{1}{p'-1} \frac{1}{v'(0)v(0)^{p'-1}} = c(p) < \infty. \end{aligned} \quad (5.13)$$

Let $A > 0$. Below we use Hölder's inequality and (5.13):

$$\int_0^A \mu(t) dt \leq \left[\int_0^A (\mu(t)v(t))^p dt \right]^{1/p} \cdot \left[\int_0^A \frac{dt}{v(t)^{p'}} \right]^{1/p'} \leq c(p) \left[\int_0^A (\mu(t)v(t))^p dt \right]^{1/p}.$$

Now, to obtain (5.12), in the last inequality we let A tend to infinity. Let us now go over to the proof of the lemma. By Theorem 2.2, the general solution of (2.4) is of the form

$$z(x) = c_1 u(x) + c_2 v(x), \quad x \in \mathbb{R}.$$

Let $z \in L_{p,\mu}$. Then $c_2 = 0$. Indeed, if $c_2 \neq 0$, then denote $x_1 \gg 1$, a number such that for all $x \geq x_1$ we have the inequality (see (2.7)):

$$\left| \frac{c_1}{c_2} \right| \frac{u(x)}{v(x)} \leq \frac{1}{2}, \quad x \geq x_1. \quad (5.14)$$

Now from (5.12), (5.14) and Theorem 2.2 it follows that

$$\begin{aligned} \infty &> \|z\|_{p,\mu}^p = \int_{-\infty}^\infty |\mu(x)(c_1 u(x) + c_2 v(x))|^p dx \\ &\geq |c_2|^p \int_{x_1}^\infty (\mu(x)v(x))^p \left| 1 - \frac{c_1}{c_2} \frac{u(x)}{v(x)} \right|^p dx \geq \left| \frac{c_2}{2} \right|^p \int_{x_1}^\infty (\mu(x)v(x))^p dx = \infty. \end{aligned}$$

We get a contradiction. Hence $c_2 = 0$. The equality $c_1 = 0$ now follows from (2.5) and (5.12). \square

Proof of Theorem 4.3 for $p \in (1, \infty)$. Necessity.

We need the following lemma.

Lemma 5.1. *Let $p \in [1, \infty)$. Suppose that conditions (4.2) hold, and the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1). Then, if $f \in L_p$ and $\text{supp } f = [x_1, x_2]$, $x_2 - x_1 < \infty$, then $f \in L_{p,\theta}$ and the solution $y \in L_{p,\mu}$ of (1.1) which corresponds to f is of the form (2.13).*

Proof. Below we only consider the case $p \in (1, \infty)$ (for $p = 1$ the arguments are similar). Let us continue the function f by zero beyond the segment $[x_1, x_2]$ and maintain the original notation. From the obvious inequalities

$$c^{-1} \leq \theta(x) \leq c, \quad x \in [x_1, x_2], \quad c = c(x_1, x_2), \quad (5.15)$$

it follows that $f \in L_{p,\theta}$. Set (see (2.9), (2.13))

$$\begin{aligned}\tilde{y}(x) &= \int_{-\infty}^{\infty} G(x, t) f(t) dt \\ &= u(x) \int_{-\infty}^x v(t) f(t) dt + v(x) \int_x^{\infty} u(t) f(t) dt, \quad x \in \mathbb{R}.\end{aligned}\quad (5.16)$$

Let us estimate the integrals in (5.16):

$$\begin{aligned}\int_{-\infty}^x v(t) |f(t)| dt &\leq \left[\int_{x_1}^{x_2} \left(\frac{v(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \cdot \left[\int_{x_1}^{x_2} |\theta(t) f(t)|^p dt \right]^{1/p} \\ &\leq c \left(\int_{x_1}^{x_2} v(t)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\theta}, \quad x \in \mathbb{R},\end{aligned}\quad (5.17)$$

$$\begin{aligned}\int_x^{\infty} u(t) |f(t)| dt &\leq \left[\int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \cdot \left[\int_{x_1}^{x_2} |\theta(t) f(t)|^p dt \right]^{1/p} \\ &\leq c \left(\int_{x_1}^{x_2} u(t)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p,\theta}, \quad x \in \mathbb{R}..\end{aligned}\quad (5.18)$$

From (5.17) and (5.18) it follows that the function $\tilde{y}(x)$, $s \in \mathbb{R}$, is well-defined. It is also easy to see that the function $\tilde{y}(x)$, $x \in \mathbb{R}$ is a particular solution of (1.1). But, since $f \in L_{p,\theta}$, (1.1) has a unique solution $y \in L_{p,\theta}$. This means that we have the equality

$$y(x) = \tilde{y}(x) + c_1 u(x) + c_2 v(x), \quad x \in \mathbb{R}.$$

Let us check that $c_1 = c_2 = 0$. Assume, say, that $c_2 \neq 0$. Then for $x \geq x_2$, we get

$$\begin{aligned}|y(x)| &\geq |c_2 v(x) - |c_1| u(x) - u(x) \int_{x_1}^{x_2} v(t) |f(t)| dt| \\ &= |c_1| v(x) \left[1 - \left| \frac{c_1}{c_2} \right| \frac{u(x)}{v(x)} - \frac{u(x)}{v(x)} \int_{x_1}^{x_2} v(t) |f(t)| dt \right].\end{aligned}$$

From (2.7) and (5.17) it follows that there exists $x_3 \geq \max\{1, x_2\}$ such that

$$|y(x)| \geq \frac{1}{2} |c_2| v(x) \quad \text{for } x \geq x_3 \quad \Rightarrow \quad (\text{see (5.12)}):$$

$$\infty > \|y\|_{p,\mu}^p \geq \int_{x_1}^{\infty} |\mu(x) y(x)|^p dx \geq \left| \frac{c_2}{2} \right|^p \int_{x_3}^{\infty} |\mu(x) v(x)|^p dx = \infty.$$

We get a contradiction. Hence $c_2 = 0$. Similarly, we prove that also $c_1 = 0$, and therefore $y = \tilde{y}$ (see (5.16)). Let $[x_1, x_2]$ be any finite segment. Set

$$f(t) = \begin{cases} \theta(t)^{-p'} \cdot u(t)^{p'-1}, & t \in [x_1, x_2] \\ 0, & t \notin [x_1, x_2] \end{cases} \quad (5.19)$$

Then

$$\|f\|_{L_{p,\theta}}^p = \int_{x_1}^{x_2} |\theta(t) f(t)|^p dt = \int_{x_1}^{x_2} \frac{\theta(t)^p u^{p(p'-1)}(t)}{\theta(t)^{p'p}} dt = \int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt < \infty. \quad (5.20)$$

Therefore, since the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), in the case of (5.19) equation (1.1) has a solution $y \in L_{p,\mu}$. This solution is of the form (2.13) (see Lemma 5.1). This implies that

$$\begin{aligned}
\infty &> \|y\|_{p,\mu}^p = \int_{-\infty}^{\infty} |\mu(x)y(x)|^p dx \\
&= \left\{ \int_{-\infty}^{\infty} \mu(x)^p \left[u(x) \int_{-\infty}^x v(t)f(t)dt + v(x) \int_x^{\infty} u(t)f(t)dt \right]^p dx \right\} \\
&\geq \int_{-\infty}^{\infty} (\mu(x)v(x))^p \left(\int_x^{\infty} u(t)f(t)dt \right)^p dx \geq \int_{-\infty}^{x_1} (\mu(x)v(x))^p \left(\int_x^{\infty} u(t)f(t)dt \right)^p dx \\
&\geq \int_{-\infty}^{x_1} (\mu(x)v(x))^p dx \left(\int_{x_1}^{x_2} u(t)f(t)dt \right)^p = \int_{-\infty}^{x_1} (\mu(x)v(x))^p dx \left(\int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^p \quad (5.21)
\end{aligned}$$

Now, using (5.21), (5.20) and (1.5), we obtain

$$\begin{aligned}
\left[\int_{-\infty}^{x_1} (\mu(x)v(x))^p dx \right]^{1/p} \int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt &\leq \|y\|_{p,\mu} \leq c(p) \|f\|_{p,\theta} \\
&= c(p) \left[\int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \Rightarrow
\end{aligned}$$

$$\left(\int_{-\infty}^{x_1} (\mu(t)v(t))^p dt \right)^{1/p} \left(\int_{x_1}^{x_2} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \leq c(p) < \infty.$$

Since in this inequality x_1 and x_2 ($x_1 \leq x_2$) are arbitrary numbers, we conclude that

$$M = \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x (\mu(t)v(t))^p dt \right)^{1/p} \cdot \left(\int_x^{\infty} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \leq c(p) < \infty.$$

This inequality means that the operator $S_2 : L_p \rightarrow L_p$,

$$(S_2 f)(x) = \mu(x)v(x) \int_x^{\infty} \frac{u(t)}{\theta(t)} f(t)dt, \quad x \in \mathbb{R} \quad (5.22)$$

is bounded (see Theorem 2.7). Similarly, we use Theorem 2.8 to conclude that the operator

$S_1 : L_p \rightarrow L_p$,

$$(S_1 f)(x) = \mu(x)u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} f(t)dt, \quad x \in \mathbb{R} \quad (5.23)$$

is bounded. Since we have the equality (see (2.9) and (4.3))

$$S = S_1 + S_2 \quad (5.24)$$

our assertion now follows from the triangle inequality for norms.

Proof of Theorem 4.3. Sufficiency.

Lemma 5.2. *Let $p \in [1, \infty)$, and let S, S_1, S_2 be operators (4.3), (5.23) and (5.22), respectively. Then we have the inequalities*

$$\frac{\|S_1\|_{p \rightarrow p} + \|S_2\|_{p \rightarrow p}}{2} \leq \|S\|_{p \rightarrow p} \leq \|S_1\|_{p \rightarrow p} + \|S_2\|_{p \rightarrow p}. \quad (5.25)$$

Proof. The upper estimate in (5.25) follows from (5.24). To prove the lower estimate in (5.25), we use the following obvious relations:

$$\begin{aligned} \|S_1(f)\|_p^p &= \int_{-\infty}^{\infty} \mu(x)^p \left| u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} f(t) dt \right|^p dx \\ &\leq \int_{-\infty}^{\infty} \mu(x)^p \left(u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} |f(t)| dt \right)^p dx \\ &\leq \int_{-\infty}^{\infty} \mu(x)^p \left[u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} |f(t)| dt + v(x) \int_x^{\infty} \frac{u(t)}{\theta(t)} |f(t)| dt \right]^p dx \\ &= \int_{-\infty}^{\infty} \left| \mu(x) \int_{-\infty}^x \frac{G(x, t)}{\theta(t)} |f(t)| dt \right|^p dx = \|S(|f|)\|_p^p \leq \|S\|_{p \rightarrow p}^p \cdot \|f\|_p^p. \end{aligned}$$

This implies that $\|S_1\|_{p \rightarrow p} \leq \|S\|_{p \rightarrow p}$. Similarly, we check that $\|S_2\|_{p \rightarrow p} \leq \|S\|_{p \rightarrow p}$. These inequalities imply the lower estimate in (5.25). \square

Let us now go over to the proof of the theorem. Since (2.1) holds, equation (2.4) has a FSS $\{u, v\}$ with the properties from Theorem 2.2. Since the operator $S : L_p \rightarrow L_p$ is bounded, so are also the operators $S_i : L_p \rightarrow L_p$, $i = 1, 2$ (see (5.25)). Then, by Theorems Theorem 2.7 and Theorem 2.8, we obtain the inequalities

$$\tilde{M}_p \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x \left(\frac{v(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \left(\int_x^{\infty} (\mu(t) u(t)^p dt) \right)^{1/p} < \infty, \quad (5.26)$$

$$M_p \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x (\mu(t) v(t)^p dt) \right)^{1/p} \cdot \left(\int_x^{\infty} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} < \infty. \quad (5.27)$$

These inequalities imply that the function

$$y(x) = (Gf)(x) = u(x) \int_{-\infty}^x v(t) f(t) dt + v(x) \int_x^{\infty} u(t) f(t) dt, \quad x \in \mathbb{R} \quad (5.28)$$

is well-defined because the integrals in (5.28) converge:

$$\begin{aligned} \int_{-\infty}^x v(t) |f(t)| dt &\leq \left(\int_{-\infty}^x \left(\frac{v(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p, \theta}, \quad x \in \mathbb{R}, \\ \int_x^{\infty} u(t) |f(t)| dt &\leq \left(\int_x^{\infty} \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \|f\|_{p, \theta}, \quad x \in \mathbb{R}. \end{aligned}$$

Further, one can check in a straightforward manner (see Theorem 2.2) that the function $y(x)$, $x \in \mathbb{R}$ is a solution of (1.1). In addition,

$$\begin{aligned} \|y\|_{p,\mu} &= \left[\int_{-\infty}^{\infty} \left(\mu(x) \left| \int_{-\infty}^{\infty} G(x,t) f(t) dt \right| \right)^p dx \right]^{1/p} \\ &= \left[\int_{-\infty}^{\infty} \left(\mu(x) \left| \int_{-\infty}^{\infty} \frac{G(x,t)}{\theta(t)} (\theta(t) f(t)) dt \right| \right)^p dx \right]^{1/p} \\ &= \|S(\theta f)\|_p \leq \|S\|_{p \rightarrow p} \cdot \|\theta f\|_p = \|S\|_{p \rightarrow p} \cdot \|f\|_{p,\theta}, \end{aligned}$$

i.e., (1.5) holds. It only remains to refer to Lemma 4.2. \square

Proof of Theorem 4.3 for $p = 1$. Necessity.

Let $[x_1, x_2]$ be an arbitrary finite segment, and let $f \in L_1$ be such that $\text{supp } f = [x_1, x_2]$. Then (see (5.15)) $f \in L_{1,\theta}$ and therefore equation (1.1) with such a right-hand side has a unique solution $y \in L_{1,\mu}$. By Lemma 5.1, this solution is given by formula (2.13) and satisfies (1.5). Let us introduce the operator \tilde{S} :

$$(\tilde{S}g)(x) = \mu(x) \int_{x_1}^{x_2} \frac{G(x,t)}{\theta(t)} g(t) dt, \quad x \in [x_1, x_2], \quad g \in L_1(x_1, x_2)$$

and the function g given on the sequence $[x_1, x_2]$ by the formula

$$g(x) = \theta(x) f(x), \quad x \in [x_1, x_2].$$

Then we have

$$\begin{aligned} \|\tilde{S}g\|_{L_1(x_1, x_2)} &= \int_{x_1}^{x_2} \left| \mu(x) \int_{x_1}^{x_2} \frac{G(x,t)}{\theta(t)} g(t) dt \right| dx \\ &= \int_{x_1}^{x_2} \mu(x) \left| \int_{x_1}^{x_2} G(x,t) f(t) dt \right| dx = \int_{x_1}^{x_2} \mu(x) \left| \int_{-\infty}^{\infty} G(x,t) f(t) dt \right| dx \\ &= \int_{x_1}^{x_2} \mu(x) |y(x)| dx \leq \int_{-\infty}^{\infty} \mu(x) |y(x)| dx = \|y\|_{1,\mu} \leq c(1) \|f\|_{1,\theta} \\ &= c(1) \int_{-\infty}^{\infty} \theta(t) |f(t)| dt = c(1) \int_{x_1}^{x_2} |\theta(t) f(t)| dt = c(1) \|g\|_{L_1(x_1, x_2)}. \end{aligned}$$

Together with (2.22) and (2.9), this implies that

$$\begin{aligned} \|\tilde{S}\|_{L_1(x_1, x_2) \rightarrow L_1(x_1, x_2)} &\leq c(1) \quad \Rightarrow \\ \sup_{x \in [x_1, x_2]} \frac{1}{\theta(x)} \int_{x_1}^{x_2} \mu(t) G(x,t) dt &= \|\tilde{S}\|_{L_1(x_1, x_2) \rightarrow L_1(x_1, x_2)} \leq c(1). \end{aligned}$$

In the last inequality, x_1 and x_2 are arbitrary numbers. Hence

$$\sup_{x \in \mathbb{R}} \frac{1}{\theta(x)} \int_{-\infty}^{\infty} \mu(t) G(x,t) dt \leq c(1) < \infty.$$

But then by Theorem 2.9 we obtain that $\|S\|_{L_1 \rightarrow L_2} \leq c(1) < \infty$, as required.

Proof of Theorem 4.3 for $p = 1$. Sufficiency.

From (2.1) it follows that equation (2.4) has a FSS $\{u, v\}$ (see Theorem 2.2), the Green function and the operator S are defined (see (2.9) and (4.3)). Further, the operators S_i , $i = 1, 2$ (see (4.4), (4.5)) are bounded because so is the operator $S : L_1 \rightarrow L_1$ (see Lemma 5.2). Let now $f \in L_{1,\theta}$ and $g = \theta \cdot |f|$. Then $0 \leq g \in L_1$, $S_i g \in L_1$, $i = 1, 2$, and one has the inequalities

$$0 \leq (S_i g)(x) < \infty, \quad \forall x \in \mathbb{R}, \quad i = 1, 2. \quad (5.29)$$

We will prove (5.29) for $i = 1$ (the case $i = 2$ is considered in a similar way). Assume to the contrary that there exists $x_1 \in \mathbb{R}$ such that $(S_1 g)(x_1) = \infty$. Let $x_2 > x_1$. Then, since the functions μ and u are continuous, we have

$$\begin{aligned} (S_1 g)(x_2) &= \mu(x_2)u(x_2) \int_{-\infty}^{x_2} \frac{v(t)}{\theta(t)} g(t) dt \\ &\geq \frac{\mu(x_2)u(x_2)}{\mu(x_1)u(x_1)} \left[\mu(x_1)u(x_1) \int_{-\infty}^{x_1} \frac{v(t)}{\theta(t)} g(t) dt \right] = \frac{\mu(x_2)u(x_2)}{\mu(x_1)u(x_1)} (S_1 g)(x_1) = \infty \\ \Rightarrow \\ \infty &> \|Sg\|_1 = \int_{-\infty}^{\infty} \mu(x)u(x) \left| \int_{-\infty}^x \frac{v(t)}{\theta(t)} g(t) dt \right| dx \\ &\geq \int_{x_1}^{\infty} \mu(x)u(x) \left(\int_{-\infty}^x \frac{v(t)}{\theta(t)} g(t) dt \right) dx = \int_{x_1}^{\infty} (S_1 g)(x) dx = \infty. \end{aligned}$$

We get a contradiction. Hence, inequalities (5.29) hold. From (5.29) and the definition of g we obtain

$$\int_{-\infty}^x v(t)|f(t)|dt < \infty, \quad \int_x^{\infty} u(t)|f(t)|dt < \infty \quad \forall x \in \mathbb{R}. \quad (5.30)$$

For instance,

$$\begin{aligned} \int_{-\infty}^x v(t)|f(t)|dt &= \frac{1}{\mu(x)u(x)} \left[\mu(x)u(x) \int_{-\infty}^x \frac{v(t)}{\theta(t)} \cdot (\theta(t)|f(t)|)dt \right] \\ &= \frac{1}{\mu(x)u(x)} (S_1 g)(x) < \infty \quad \Rightarrow \quad (5.30) \end{aligned}$$

Thus, if $f \in L_{1,\theta}$, then by (5.30) the following integrals converge:

$$\int_{-\infty}^x v(t)f(t)dt, \quad \int_x^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R}$$

and therefore, for $x \in \mathbb{R}$, the function

$$y(x) = (Gf)(x) = u(x) \int_{-\infty}^x v(t)f(t)dt + v(x) \int_x^{\infty} u(t)f(t)dt, \quad x \in \mathbb{R}$$

is well-defined. This immediately implies that $y(x)$ is a solution of (1.1). In addition, (1.5) holds:

$$\begin{aligned} \|\mu y\|_1 &= \int_{-\infty}^{\infty} \mu(x) \left| \int_{-\infty}^{\infty} \frac{G(x,t)}{\theta(t)} (\theta(t)f(t)) dt \right| dx \leq \int_{-\infty}^{\infty} \mu(x) \int_{-\infty}^{\infty} \frac{G(x,t)}{\theta(t)} |g(t)| dt dx \\ &= \|Sg\|_1 \leq \|S\|_{1 \rightarrow 1} \cdot \|g\|_1 = \|S\|_{1 \rightarrow 1} \cdot \|f\|_{1,\theta} \Rightarrow (1.5). \end{aligned}$$

It remains to note that by Lemma 4.2 this solution is unique in the class $L_{1,\mu}$. \square

Proof of Lemma 4.5. From (2.2) we obtain the inequality

$$2 \leq \sqrt{2}d(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi, \quad x \in \mathbb{R}.$$

Together with the formula for $|d'(x)|$ (see the proof of Lemma 3.1), this implies that

$$\begin{aligned} |d'(x)| &\leq \frac{d(x)}{\sqrt{2}} \left| \int_x^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^x q(\xi) d\xi \right| \cdot \left(d(x) \int_{x-\sqrt{2}d(x)}^{x+\sqrt{2}d(x)} q(\xi) d\xi \right)^{-1} \\ &\leq \frac{1}{2}d(x) \left| \int_x^{x+\sqrt{2}d(x)} q(\xi) d\xi - \int_{x-\sqrt{2}d(x)}^x q(\xi) d\xi \right| = \frac{\nu(x)}{2}, \quad x \in \mathbb{R} \Rightarrow \\ &\lim_{|x| \rightarrow \infty} d'(x) = 0. \end{aligned} \tag{5.31}$$

Let us now go to (4.8). Fix $\varepsilon \in (0, 1/\sqrt{2})$ (see (4.9) regarding the case $\varepsilon \geq 1/\sqrt{2}$). Then there exists $x_0 = x_0(\varepsilon) \gg 1$ such that we have the inequality (see (5.31))

$$|d'(x)| \leq \varepsilon \quad \text{if} \quad |x| \geq x_0. \tag{5.32}$$

It is easy to see that all possible cases of placing the numbers $t, x \in \mathbb{R}$ and the segments $(-\infty, x_0]$, $[-x_0, x_0]$ and $[x_0, \infty]$ can be put in the following table:

1.1 $x \in (-\infty, -x_0]$ $t \in (-\infty, -x_0]$	1.2 $x \in (-\infty, -x_0]$ $t \in [-x_0, x_0]$	1.3 $x \in (-\infty, -x_0]$ $t \in [x_0, \infty]$
2.1 $x \in [-x_0, x_0]$ $t \in (-\infty, -x_0]$	2.2 $x \in [-x_0, x_0]$ $t \in [-x_0, x_0]$	2.3 $x \in [-x_0, x_0]$ $x \in [x_0, \infty)$
3.1 $x \in (x_0, \infty]$ $t \in (-\infty, -x_0]$	3.2 $x \in (x_0, \infty)$ $t \in [-x_0, x_0]$	3.3 $x \in (x_0, \infty)$ $t \in [-x_0, \infty)$

(5.33)

We check inequalities (4.8) separately in each case appearing in (5.33).

Cases 1.1 and 3.3.

Both cases are treated in the same way. Let us introduce the standing notation for the whole proof:

$$\begin{aligned} m(\varepsilon) &= \min_{t \in [-x_0, x_0]} d(t), & M(\varepsilon) &= \max_{t \in [-x_0, x_0]} d(t) \\ c(\varepsilon) &= \max \left\{ \frac{1}{m(\varepsilon)}, M(\varepsilon) \right\}, \\ a &= \min\{x, t\}, & b &= \max\{x, t\}. \end{aligned}$$

Consider, say, Case 3.3. The following implications are obvious:

$$\begin{aligned} -\varepsilon \leq d'(\xi) \leq \varepsilon \quad \text{for} \quad \xi \in [a, b] &\Rightarrow -\frac{\varepsilon}{d(\xi)} \leq \frac{d'(\xi)}{d(\xi)} \leq \frac{\varepsilon}{d(\xi)}, \quad \xi \in [a, b] \\ \Rightarrow -\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| &= -\varepsilon \int_a^b \frac{d\xi}{d(\xi)} \leq \ln \frac{d(b)}{d(a)} \leq \varepsilon \int_a^b \frac{d\xi}{d(\xi)} = \varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \Rightarrow \\ \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) &\leq \frac{d(b)}{d(a)}, \quad \frac{d(a)}{d(b)} \leq \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \Rightarrow \quad (4.8). \end{aligned}$$

Cases 1.2 and 2.1.

Both cases are treated in the same way. For instance, in Case 1.2 we have

$$\begin{aligned} \frac{d(t)}{d(x)} &= \frac{d(t)}{d(-x_0)} \cdot \frac{d(-x_0)}{d(x)} \leq c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} \right| \right) \\ &\leq c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) = c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right); \\ \frac{d(t)}{d(x)} &= \frac{d(t)}{d(-x_0)} \cdot \frac{d(-x_0)}{d(x)} \geq c(\varepsilon)^{-2} \exp \left(-\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} \right| \right) \\ &\geq c(\varepsilon)^2 \exp \left(-\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^t \frac{d\xi}{d(\xi)} \right| \right) = c(\varepsilon)^2 \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \Rightarrow \quad (4.8). \end{aligned}$$

Cases 1.3 and 3.1.

Both cases are treated in the same way. For instance, in Case 1.3 we have

$$\begin{aligned} \frac{d(t)}{d(x)} &= \frac{d(-x_0)}{d(x)} \cdot \frac{d(x_0)}{d(-x_0)} \cdot \frac{d(t)}{d(x_0)} \leq \frac{M}{m} \exp \left(\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} \right| + \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \\ &\leq c(\varepsilon)^2 \exp \left[\varepsilon \left(\int_x^{-x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^x \frac{d\xi}{d(\xi)} \right) + \varepsilon \int_{x_0}^t \frac{d\xi}{d(\xi)} \right] \\ &= c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right); \end{aligned}$$

$$\begin{aligned} \frac{d(t)}{d(x)} &= \frac{d(-x_0)}{d(x)} \cdot \frac{d(x_0)}{d(-x_0)} \cdot \frac{d(t)}{d(x_0)} \geq \frac{m}{M} \exp \left(-\varepsilon \left| \int_x^{-x_0} \frac{d\xi}{d(\xi)} \right| - \varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \\ &\geq c(\varepsilon)^{-2} \exp \left[-\varepsilon \left(\int_x^{-x_0} \frac{d\xi}{d(\xi)} + \int_{-x_0}^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)} \right) \right] \geq c(\varepsilon)^{-2} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \end{aligned}$$

Case 2.2.

We have

$$\begin{aligned} \frac{d(t)}{d(x)} &\leq \frac{M(\varepsilon)}{m(\varepsilon)} \leq c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right); \\ \frac{d(t)}{d(x)} &\geq \frac{m(\varepsilon)}{M(\varepsilon)} \geq c(\varepsilon)^{-2} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \end{aligned}$$

Cases 2.3 and 3.2.

Both cases are treated in the same way. For instance, in Case 2.3 we have

$$\begin{aligned} \frac{d(t)}{d(x)} &= \frac{d(x_0)}{d(x)} \cdot \frac{d(t)}{d(x_0)} \leq \frac{M(\varepsilon)}{m(\varepsilon)} \exp \left(\varepsilon \left| \int_{x_0}^t \frac{d\xi}{d(\xi)} \right| \right) \\ &\leq c(\varepsilon)^2 \exp \left[\varepsilon \left(\int_x^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)} \right) \right] = c(\varepsilon)^2 \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right); \end{aligned}$$

$$\begin{aligned} \frac{d(t)}{d(x)} &= \frac{d(x_0)}{d(x)} \cdot \frac{d(t)}{d(x_0)} \geq \frac{m(\varepsilon)}{M(\varepsilon)} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \\ &\geq c(\varepsilon)^{-2} \exp \left[-\varepsilon \left(\int_x^{x_0} \frac{d\xi}{d(\xi)} + \int_{x_0}^t \frac{d\xi}{d(\xi)} \right) \right] = c(\varepsilon)^{-2} \exp \left(-\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right). \end{aligned}$$

□

Proof of Theorem 4.7 for $p \in (1, \infty)$. Necessity.

We need some auxiliary assertions.

Lemma 5.3. *Let $p \in [1, \infty)$, $p' = p(p-1)^{-1}$. Denote*

$$M_p(x) = \left(\int_{-\infty}^x (\mu(t)v(t))^p dt \right)^{1/p} \cdot \left(\int_x^\infty \left(\frac{u(t)}{\theta(t)} \right)^{1/p'} dt \right), \quad x \in \mathbb{R}, \quad (5.34)$$

$$\tilde{M}_p(x) = \left(\int_{-\infty}^x \left(\frac{v(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \cdot \left(\int_x^\infty (\mu(t)u(t))^p dt \right)^{1/p}, \quad x \in \mathbb{R}. \quad (5.35)$$

Then we have the equalities (see (2.8)):

$$M_p(x) = \left[\int_{-\infty}^x \left(\sqrt{\rho(t)} \mu(t) \right)^p \exp \left(-\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \cdot \left[\int_x^\infty \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'}, \quad x \in \mathbb{R}, \quad (5.36)$$

$$\tilde{M}_p(x) = \left[\int_{-\infty}^x \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \cdot \left[\int_x^\infty \left(\mu(t) \sqrt{\rho(t)} \right)^p \exp \left(-\frac{p}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p}, \quad x \in \mathbb{R}. \quad (5.37)$$

Proof. Equalities (5.36) and (5.37) are proved in the same way. Consider, say, (5.36). This equality can be obtained by substituting formulas (2.10) in (5.34):

$$\begin{aligned} M_p(x) &= \left[\int_{-\infty}^x \left(\mu(t) \sqrt{\rho(t)} \right)^p \exp \left(\frac{p}{2} \int_{x_0}^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[\int_x^\infty \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_{x_0}^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \\ &= \left[\int_{-\infty}^x \left(\mu(t) \sqrt{\rho(t)} \right)^p \exp \left(-\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) \cdot \exp \left(\frac{p}{2} \int_{x_0}^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[\int_x^\infty \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) \cdot \exp \left(-\frac{p'}{2} \int_{x_0}^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \\ &= \left[\int_{-\infty}^x \left(\mu(t) \sqrt{\rho(t)} \right)^p \exp \left(-\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[\int_x^\infty \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'}. \end{aligned}$$

□

Let us introduce some more notation:

$$\varphi(x, t) = \begin{cases} \frac{\mu(x)v(x)}{\mu(t)v(t)}, & \text{if } x \leq t \\ \frac{\mu(t)v(t)}{\mu(x)v(x)}, & \text{if } x \geq t \end{cases}, \quad \psi(x, t) = \begin{cases} \frac{\theta(x)u(t)}{\theta(t)u(x)}, & \text{if } x \leq t \\ \frac{\theta(t)u(x)}{\theta(x)u(t)}, & \text{if } x \geq t \end{cases}. \quad (5.38)$$

Lemma 5.4. *Under the hypotheses of the theorem, for a given $\varepsilon > 0$ and for all $t, x \in \mathbb{R}$, we have the inequality*

$$\max\{\varphi(x, t); \psi(x, t)\} \leq c(\varepsilon) \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right). \quad (5.39)$$

Proof. We will check inequality (5.39) for the function φ (for the function ψ the proof of (5.39) is similar). Below we use (2.10), (2.12) and (4.10). Let $x \geq t$. Then

$$\begin{aligned} \frac{\mu(t)}{\mu(x)} \cdot \frac{v(t)}{v(x)} &= \frac{\mu(t)}{\mu(x)} \cdot \sqrt{\frac{\rho(t)}{\rho(x)}} \exp \left(-\frac{1}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) \leq c \frac{\mu(t)}{\mu(x)} \cdot \sqrt{\frac{d(t)}{d(x)}} \exp \left(-\frac{1}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) \\ &\leq c(\varepsilon) \exp \left(\varepsilon \int_t^x \frac{d\xi}{d(\xi)} - \frac{1}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) \leq c(\varepsilon) \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \int_t^x \frac{d\xi}{\rho(\xi)} \right) \\ &= c(\varepsilon) \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right); \end{aligned}$$

Similarly, for $x \leq t$, we have:

$$\begin{aligned} \frac{\mu(x)}{\mu(t)} \cdot \frac{v(x)}{v(t)} &= \frac{\mu(x)}{\mu(t)} \sqrt{\frac{\rho(x)}{\rho(t)}} \exp \left(-\frac{1}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) \leq c \frac{\mu(x)}{\mu(t)} \sqrt{\frac{d(x)}{d(t)}} \exp \left(-\frac{1}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) \\ &\leq c(\varepsilon) \exp \left(\varepsilon \int_t^x \frac{d\xi}{d(\xi)} - \frac{1}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) \leq c(\varepsilon) \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \int_t^x \frac{d\xi}{\rho(\xi)} \right) \\ &= c(\varepsilon) \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right). \end{aligned}$$

□

Lemma 5.5. *Under conditions (1.1) and (2.1), we have the inequality*

$$\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \leq 8, \quad \forall x \in \mathbb{R}. \quad (5.40)$$

Proof. Estimate (5.40) follows from (3.2):

$$\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} = \int_{x-d(x)}^{x+d(x)} \frac{d(x)}{d(\xi)} \cdot \frac{d\xi}{d(x)} \leq \int_{x-d(x)}^{x+d(x)} 4 \frac{d\xi}{d(x)} = 8.$$

□

Lemma 5.6. *Under the hypotheses of the theorem, we have the inequalities*

$$c^{-1} \leq \frac{\mu(t)}{\mu(x)}, \quad \frac{\theta(t)}{\theta(x)} \leq c; \quad \text{if } t \in [x - d(x), x + d(x)], \quad x \in \mathbb{R}. \quad (5.41)$$

Proof. We will only check inequalities (5.41) for the function μ (the proof of (5.41) for the function θ is similar). In (4.10), set $\varepsilon = \frac{1}{2}$. Now for $|t - x| \leq d(x)$, $x \in \mathbb{R}$, we use (3.2), (4.10)

and (5.40):

$$\begin{aligned}\frac{\mu(t)}{\mu(x)} &\leq c \sqrt{\frac{d(x)}{d(t)}} \exp \left(\frac{1}{2} \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \leq c \exp \left(\frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \leq c < \infty, \\ \frac{\mu(t)}{\mu(x)} &\geq c^{-1} \sqrt{\frac{d(x)}{d(t)}} \exp \left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{d(\xi)} \right| \right) \geq c^{-1} \exp \left(-\frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) \geq c^{-1} > 0.\end{aligned}$$

□

Let us now go over to the theorem. Since condition (2.1) holds, by Theorem 2.2, a FSS $\{u, v\}$ of equation (2.4) is defined, and thus the operator S (see (4.3)) is also defined. Since the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1), by Theorem 4.3 the operator $S : L_p \rightarrow L_p$, $p \in [1, \infty)$ is bounded. Then so are the operators $S_i : L_p \rightarrow L_p$, $i = 1, 2$ (see (5.25)). Let $p \in (1, \infty)$. Consider, say, the operator $S_2 : L_p \rightarrow L_p$. Since it is bounded, we have $M_p < \infty$ by Theorem 2.7 (see (5.27) and (5.34)). Below we use this fact together with Lemma 2.1,

(2.12), (5.40) and (5.41):

$$\begin{aligned}
\infty &> M_p = \sup_{x \in \mathbb{R}} M_p(x) = \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x (\mu(t)v(t))^p dt \right)^{1/p} \left(\int_x^\infty \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right)^{1/p'} \\
&= \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x \left(\sqrt{\rho(t)\mu(t)} \right)^p \exp \left(-\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^\infty \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \\
&\geq \sup_{x \in \mathbb{R}} \left[\int_{x-d(x)}^x \left(\sqrt{\rho(t)\mu(t)} \right)^p \exp \left(-\frac{p}{2} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^{x+d(x)} \left(\frac{\sqrt{\rho(t)}}{\theta(t)} \right)^{p'} \exp \left(-\frac{p'}{2} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \left[\int_{x-d(x)}^x \left(\sqrt{d(t)\mu(t)} \right)^p \exp \left(-\sqrt{2}p \int_t^x \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^{x+d(x)} \left(\frac{\sqrt{d(t)}}{\theta(t)} \right)^{p'} \exp \left(-\sqrt{2}p' \int_x^t \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p'} \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \left[\int_{x-d(x)}^x \left(\sqrt{d(t)\mu(t)} \right)^p \exp \left(-\sqrt{2}p \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^{x+d(x)} \left(\frac{\sqrt{d(t)}}{\theta(t)} \right)^{p'} \exp \left(-\sqrt{2}p' \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)} \right) dt \right]^{1/p'} \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\mu(x)}{\theta(x)} d^2(x) = c^1 m(q, \mu, \theta),
\end{aligned}$$

as required. Let $p = 1$. Since the operator $S : L_1 \rightarrow L_1$ is bounded (Theorem 4.3), so are the operators $S_i : L_1 \rightarrow L_1$, $i = 1, 2$ (see Lemma 5.2). Let, say, $i = 2$. Below we use Theorem 2.9,

(5.22), Lemma 2.1, (2.10), (2.12), (5.40), (5.41) and (3.2):

$$\begin{aligned}
\infty &> \|S_2\|_{1 \rightarrow 1} = \sup_{x \in \mathbb{R}} \frac{u(x)}{\theta(x)} \int_{-\infty}^x \mu(t)v(t)dt \geq \sup_{x \in \mathbb{R}} \frac{u(x)}{\theta(x)} \int_{x-d(x)}^x \mu(t)v(t)dt \\
&= \sup_{x \in \mathbb{R}} \frac{\sqrt{\rho(x)}}{\theta(x)} \int_{x-d(x)}^x \mu(t) \sqrt{\rho(t)} \exp\left(-\frac{1}{2} \int_t^x \frac{d\xi}{\rho(\xi)}\right) \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\sqrt{d(x)}}{\theta(x)} \int_{x-d(x)}^x \mu(t) \sqrt{d(t)} \exp\left(-\sqrt{2} \int_t^x \frac{d\xi}{d(\xi)}\right) dt \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\sqrt{d(x)}}{\theta(x)} \int_{x-d(x)}^x \mu(t) \sqrt{d(t)} \exp\left(-\sqrt{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{d(\xi)}\right) dt \\
&\geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\sqrt{d(x)}}{\theta(x)} \int_{x-d(x)}^x \mu(t) \sqrt{d(t)} dt \geq c^{-1} \sup_{x \in \mathbb{R}} \frac{\mu(x)}{\theta(x)} d^2(x) = c^{-1} m(q, \mu, \theta).
\end{aligned}$$

Proof of Theorem 4.7. Sufficiency.

It is enough to show that the operators $S_i : L_p \rightarrow L_p$, $p \in [1, \infty)$, $i = 1, 2$, are bounded. Indeed, then so is the operator $S : L_p \rightarrow L_p$, $p \in [1, \infty)$ (see (5.25)), and then by Theorem 4.3 the pair $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible for (1.1). Both operators S_i , $i = 1, 2$, are treated in the same way, and therefore below we only consider the operator S_2 (see (4.5), (5.22)). Below, when estimating $\|S_2\|_{p \rightarrow p}$, $p \in (1, \infty)$, we use Theorem 2.7, (5.22), (5.34), (5.38), (5.39), (4.10) for $\varepsilon = 1/4\sqrt{2}$, (2.12) and (4.11):

$$\begin{aligned}
\|S_2\|_{p \rightarrow p} &\leq c(p) \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x (\mu(t)v(t))^p dt \right]^{1/p} \cdot \left[\int_x^\infty \left(\frac{u(t)}{\theta(t)} \right)^{p'} dt \right]^{1/p'} \\
&= c(\varepsilon) \sup_{x \in \mathbb{R}} (\mu(x)v(x))^{1/p'} \left[\int_{-\infty}^x \left(\frac{\mu(t)v(t)}{\mu(x)v(x)} \right)^{p-1} (\mu(t)v(t)) dt \right]^{1/p} \\
&\quad \cdot \left(\frac{u(x)}{\theta(x)} \right)^{1/p} \left[\int_x^\infty \left(\frac{u(t)}{\theta(t)} \cdot \frac{\theta(x)}{u(x)} \right)^{p'-1} \left(\frac{u(t)}{\theta(t)} \right) dt \right]^{1/p'} \\
&= c(\varepsilon) \sup_{x \in \mathbb{R}} \left[\frac{u(x)}{\theta(x)} \int_{-\infty}^x \varphi(x, t)^{p-1} (\mu(t)v(t)) dt \right]^{1/p} \\
&\quad \cdot \left[\mu(x)v(x) \int_x^\infty \psi(x, t)^{p'-1} \left(\frac{u(t)}{\theta(t)} \right) dt \right]^{1/p'} \\
&\leq c(\varepsilon) \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x \left(\frac{u(x)}{\theta(x)} \cdot \frac{\theta(t)}{u(t)} \right) \cdot \varphi(x, t)^{p-1} \frac{\mu(t)}{\theta(t)} \rho(t) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^\infty \left(\frac{\mu(x)v(x)}{\mu(t)v(t)} \right) \cdot \psi(x, t)^{p'-1} \cdot \frac{\mu(t)}{\theta(t)} \rho(t) dt \right]^{1/p'} \\
&= c(\varepsilon) \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x \psi(x, t) \cdot \varphi(x, t)^{p-1} \frac{\mu(t)}{\theta(t)} \rho(t) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^\infty \varphi(x, t) \psi(x, t)^{p'-1} \frac{\mu(t)}{\theta(t)} \rho(t) dt \right]^{1/p'} \\
&\leq c(\varepsilon) \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x \left(\frac{\mu(t)}{\theta(t)} d^2(t) \right) \cdot \left(\frac{\rho(t)}{d(t)} \right)^2 \cdot \frac{1}{\rho(t)} \exp \left(\left(\sqrt{2}\varepsilon - \frac{1}{2} \right) p \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^\infty \left(\frac{\mu(t)}{\theta(t)} d^2(t) \right) \cdot \left(\frac{\rho(t)}{d(t)} \right)^2 \frac{1}{\rho(t)} \exp \left(\left(\sqrt{2}\varepsilon - \frac{1}{2} \right) p' \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \\
&\leq c(\varepsilon) m(q, \mu, \theta) \sup_{x \in \mathbb{R}} \left[\int_{-\infty}^x \frac{1}{\rho(t)} \exp \left(-\frac{p}{4} \int_t^x \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[\int_x^\infty \frac{1}{\rho(t)} \exp \left(-\frac{p'}{4} \int_x^t \frac{d\xi}{\rho(\xi)} \right) dt \right]^{1/p'} \leq cm(q, \mu, \theta) < \infty.
\end{aligned}$$

Consider the case $p = 1$. Below, when estimating $\|S\|_{1 \rightarrow 1}$, we use (2.21), (4.3), (2.11) and (4.10) for $\varepsilon = 1/4\sqrt{2}$, and (2.12):

$$\begin{aligned}
\|S\|_{1 \rightarrow 1} &= \sup_{x \in \mathbb{R}} \frac{1}{\theta(x)} \int_{-\infty}^{\infty} \mu(t) G(x, t) dt = \sup_{x \in \mathbb{R}} \frac{\sqrt{\rho(x)}}{\theta(x)} \int_{-\infty}^{\infty} \mu(t) \sqrt{\rho(t)} \exp \left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) dt \\
&= \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \left(\frac{\mu(t)}{\theta(t)} d^2(t) \right) \cdot \left(\frac{\rho(t)}{d(t)} \right)^2 \frac{\theta(t) \sqrt{d(x)}}{\theta(x) \sqrt{d(t)}} \cdot \sqrt{\frac{\rho(x)}{d(x)} \cdot \frac{d(t)}{\rho(t)}} \cdot \frac{1}{\rho(t)} \exp \left(-\frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) dt \\
&\leq cm(q, \mu, \theta) \int_{-\infty}^{\infty} \frac{1}{\rho(t)} \exp \left(\varepsilon \left| \int_x^t \frac{d\xi}{d(\xi)} \right| - \frac{1}{2} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) dt \\
&\leq cm(q, \mu, \theta) \int_{-\infty}^{\infty} \frac{1}{\rho(t)} \exp \left(\left(\sqrt{2\varepsilon} - \frac{1}{2} \right) \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) dt \\
&= cm(q, \mu, \theta) \int_{-\infty}^{\infty} \frac{1}{\rho(t)} \exp \left(-\frac{1}{4} \left| \int_x^t \frac{d\xi}{\rho(\xi)} \right| \right) dt = cm(q, r, \mu) < \infty.
\end{aligned}$$

Thus the operator $S : L_p \rightarrow L_p$, $p \in [1, \infty)$ is bounded, and it remains to refer to Theorem 4.3.

Proof of Lemma 4.8. Fix $\varepsilon > 0$ and choose $x_0 = x_0(\varepsilon) \gg 1$ in order to have the inequalities

$$-\frac{\varepsilon}{3} \leq \frac{\mu'(\xi)}{\mu(\xi)} d(\xi); \quad d'(\xi) \leq \frac{\varepsilon}{3} \quad \text{for all } |\xi| \geq x_0. \quad (5.42)$$

From (5.42), one can easily deduce the estimates

$$-\frac{2}{3} \leq \frac{(\mu(\xi)d(\xi))'}{\mu(\xi)d(\xi)} \leq \frac{2\varepsilon}{3} \cdot \frac{1}{d(\xi)} \quad \text{for all } |\xi| \geq x_0. \quad (5.43)$$

Let, say, $t \geq x \geq x_0$. Then from (5.43), we obtain

$$\exp \left(-\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)} \right) \leq \frac{\mu(t)d(t)}{\mu(x)d(x)} \leq \exp \left(\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)} \right), \quad t \geq x \geq x_0. \quad (5.44)$$

Let us write (5.44) in a different way:

$$\sqrt{\frac{d(x)}{d(t)}} \exp \left(-\frac{2\varepsilon}{\varepsilon} \int_x^t \frac{d\xi}{d(\xi)} \right) \leq \frac{\mu(t)}{\mu(x)} \sqrt{\frac{d(t)}{d(x)}} \leq \sqrt{\frac{d(x)}{d(t)}} \exp \left(\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)} \right), \quad t \geq x \geq x_0.$$

We now combine the latter estimates with inequalities (4.8) written for $\frac{2\varepsilon}{3}$ instead of ε :

$$c \left(\frac{2\varepsilon}{3} \right)^{-1/2} \exp \left(-\frac{2\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)} \right) \leq \sqrt{\frac{d(x)}{d(t)}} \leq c \left(\frac{2\varepsilon}{3} \right)^{1/2} \exp \left(\frac{\varepsilon}{3} \int_x^t \frac{d\xi}{d(\xi)} \right).$$

We easily obtain that for $t \geq x \geq x_0$ we have the inequalities

$$c \left(\frac{2}{3} \varepsilon \right)^{-1/2} \exp \left(-\varepsilon \int_x^t \frac{d\xi}{d(\xi)} \right) \leq \frac{\mu(t)}{\mu(x)} \sqrt{\frac{d(t)}{d(x)}} \leq c \left(\frac{2\varepsilon}{3} \right)^{1/2} \exp \left(\varepsilon \int_x^t \frac{d\xi}{d(\xi)} \right),$$

as required. The cases $x \geq t \geq x_0$ and the cases $t \leq x \leq -x_0$, $x \leq t \leq -x_0$ are considered in a similar way. We then continue the proof as in Lemma 4.5, with obvious modifications, similar to those presented above. \square

6. EXAMPLE

In this final section, we consider equation (1.1) with

$$q(x) = \frac{1}{\sqrt{1+x^2}} + \frac{\cos(e^{|x|})}{\sqrt{1+x^2}}, \quad x \in \mathbb{R}. \quad (6.1)$$

Using the results obtained above, we show that the following assertions hold:

- A) Equation (1.1) in the case of (6.1) is not correctly solvable in L_p , for any $p \in [1, \infty)$;
- B) For equation (1.1) in the case of (6.1), for any $p \in [1, \infty)$, the following pair of spaces $\{L_{p,\mu}; L_{p,\theta}\}$ is admissible, where

$$\mu(x) = \frac{1}{\sqrt{1+x^2} \ln(2+x^2)}, \quad \theta(x) = \frac{1}{\ln(2+x^2)}, \quad x \in \mathbb{R}. \quad (6.2)$$

Remark 6.1. Below we present an algorithm for the study of (1.1) for a given pair of spaces (cases (6.1) and $\{L_p, L_p\}$ and $\{L_{p,\mu}; L_{p,\theta}\}$ in the case of (6.2)). We do not consider the question of the description of all pairs of spaces admissible for (1.1) in the case of (6.1).

For the reader's convenience, we enumerate the main steps of the proof of assertions A) and B). Note that since the functions in (6.1) and (6.2) are even, all proofs are only given for $x \in [0, \infty)$ or for $x \in [x_0, \infty)$, $x_0 \gg 1$.

1) *Checking condition (2.1).*

Let us check that in the case of (6.1) condition (2.1) holds. Assume to the contrary that there is $x_0 \in \mathbb{R}$ such that

$$\int_{x_0}^{\infty} q(t) dt = 0. \quad (6.3)$$

The function q in (6.1) is continuous and non-negative. Therefore, from (6.3) it follows that $q(t) \equiv 0$ for $t \in [x_0, \infty)$ which is obviously false. This contradiction implies (2.1).

2) *Existence of the function $d(x)$, $x \in \mathbb{R}$, and its estimates.*

From 1) and Lemma 2.1, it follows that the function $d(x)$ is defined for all $x \in \mathbb{R}$. To obtain its estimates, we use Theorem 3.7. Denote (see (3.7) and (3.8))

$$q_1(x) = \frac{1}{\sqrt{1+x^2}}; \quad q_2(x) = \frac{\cos(e^{|x|})}{\sqrt{1+x^2}}, \quad x \in \mathbb{R}; \quad (6.4)$$

$$A(x) = [0, 2\sqrt[4]{1+x^2}]; \quad \omega(x) = [x - 2\sqrt[4]{1+x^2}, x + 2\sqrt[4]{1+x^2}], \quad x \in \mathbb{R}. \quad (6.5)$$

Let us check (3.11) for the function \varkappa_1 (see (3.9)):

$$\begin{aligned}\varkappa_1(x) &= \frac{1}{q_1(x)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_1''(\xi) d\xi \right| = (1+x^2)^{3/4} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \left(\frac{1}{\sqrt{1+\xi^2}} \right)'' d\xi \right| \\ &= (1+x^2)^{3/4} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{1-2\xi^2}{1+\xi^2} \cdot \frac{d\xi}{(1+\xi^2)^{3/2}} \right|.\end{aligned}\quad (6.6)$$

Note the obvious inequalities

$$\left| \frac{1-2\xi^2}{1+\xi^2} \right| \leq \frac{1+2\xi^2}{1+\xi^2} \leq 2, \quad \xi \in \mathbb{R}. \quad (6.7)$$

In addition, for $\xi \in A(x)$, $x \gg 1$, we have

$$\frac{1+\xi^2}{1+x^2} \leq 1 + \frac{|\xi-x||\xi+x|}{1+x^2} \leq 1 + c \frac{x^{3/2}}{1+x^2} \leq 2; \quad (6.8)$$

$$\frac{1+\xi^2}{1+x^2} \geq 1 - \frac{|\xi-x||\xi+x|}{1+x^2} \geq 1 - c \frac{x^{3/2}}{1+x^2} \geq \frac{1}{2}. \quad (6.9)$$

From (6.6), (6.7), (6.8) and (6.9), it now follows that

$$\begin{aligned}\varkappa_1(x) &\leq (1+x^2)^{3/4} \sup_{t \in A(x)} \left[\int_{x-t}^{x+t} \left| \frac{1-2\xi^2}{1+\xi^2} \right| \frac{1}{(1+x^2)^{3/2}} \left(\frac{1+x^2}{1+\xi^2} \right)^{3/2} d\xi \right] \\ &\leq c \frac{(1+x^2)^{3/4}}{(1+x^2)^{3/2}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} 1 d\xi \right| = \frac{c}{\sqrt{1+x^2}} \rightarrow 0, \quad x \rightarrow \infty.\end{aligned}$$

Let us now check (3.11) for $\varkappa_2(x)$, $x \gg 1$. First show that for $x \gg 1$ we have the inequality (see (6.5)):

$$\sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{\cos e^t}{\sqrt{1+t^2}} \right| \leq c \frac{e^{-x/2}}{\sqrt{1+x^2}}, \quad x \gg 1. \quad (6.10)$$

We need the following simple assertions, given without proof:

- a) $x - \sqrt[4]{1+x^2} \rightarrow \infty$ as $x \rightarrow \infty$;
- b) the function $\varphi(\xi)$ where

$$\varphi(\xi) = \frac{e^{-\xi}}{\sqrt{1+\xi^2}}, \quad \xi \in \mathbb{R}$$

is monotone decreasing for all $\xi \in \mathbb{R}$.

Let t be any point in the interval (α, β) . Below we use assertions a), b) and the second mean theorem (see [15]):

$$\begin{aligned}\sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{\cos e^{\xi}}{\sqrt{1+\xi^2}} d\xi \right| &= \sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_{\alpha}^{\beta} \frac{e^{-\xi}}{\sqrt{1+\xi^2}} (e^{\xi} \cos e^{\xi}) d\xi \right| \\ &= \sup_{[\alpha, \beta] \subseteq \omega(x)} \frac{e^{-\alpha}}{\sqrt{1+\alpha^2}} \left| \int_{\alpha}^t e^{\xi} \cos e^{\xi} d\xi \right| \leq c \frac{e^{-\xi}}{\sqrt{1+\xi^2}} \Big|_{\xi=x-2\sqrt[4]{1+x^2}} \leq c \frac{e^{-x/2}}{\sqrt{1+x^2}}.\end{aligned}\quad (6.11)$$

Now, from (6.11) for $x \gg 1$ we obtain

$$\begin{aligned} \varkappa_2(x) &= \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(\xi) d\xi \right| = \sqrt[4]{1+x^2} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} \frac{\cos e^\xi}{\sqrt{1+\xi^2}} d\xi \right| \\ &\leq \sqrt[4]{1+x^2} \sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_\alpha^\beta \frac{\cos e^\xi}{\sqrt{1+\xi^2}} d\xi \right| \leq c \frac{e^{-x/2}}{\sqrt{1+x^2}} \Rightarrow (3.11). \end{aligned}$$

Since (3.11) is proven, by Theorem 3.7 we obtain

$$d(x) = \sqrt[4]{1+x^2}(1 + \varepsilon(x)), \quad |\varepsilon(x)| \leq 2(\varkappa_1(x) + \varkappa_2(x)), \quad |x| \gg 1, \quad (6.12)$$

$$c^{-1} \sqrt[4]{1+x^2} \leq d(x) \leq c \sqrt[4]{1+x^2}, \quad x \in \mathbb{R}. \quad (6.13)$$

3) *Proof of assertion A).*

From (6.13), it follows that $d_0 = \infty$ (see (2.3) and (2.14)). It remains to refer to Theorem 2.6. \square

Let us now go to assertion B).

4) *Checking the inclusion $q \in H$.*

To prove (4.6), we need estimates of $\tau_1(x)$ and $\tau_2(x)$ for $x \gg 1$ where (see (6.1) and (6.4))

$$\tau_1(x) = \left| \int_0^{\sqrt{2}d(x)} (q_1(x+t) - q_1(x-t)) dt \right|; \quad (6.14)$$

$$\tau_2(x) = \left| \int_0^{\sqrt{2}d(x)} (q_2(x+t) - q_2(x-t)) dt \right|. \quad (6.15)$$

To estimate $\tau_1(x)$, we use below (6.7), (6.8), (6.9) and (6.12):

$$\begin{aligned} \tau_1(x) &= \left| \int_0^{\sqrt{2}d(x)} \left(\int_{x-t}^{x+t} q'_1(\xi) d\xi \right) dt \right| \leq \sqrt{2}d(x) \sup_{|t| \leq \sqrt{2}d(x)} \left| \int_{x-\xi}^{x+\xi} q'_1(t) dt \right| \\ &\leq c \sqrt[4]{1+x^2} \sup_{|\xi| \leq 2\sqrt[4]{1+t^2}} \left| \int_{x-\xi}^{x+\xi} \frac{t}{\sqrt{1+t^2}} \cdot \frac{1+x^2}{1+t^2} \cdot \frac{dt}{1+x^2} \right| \\ &\leq c \frac{\sqrt[4]{1+x^2}}{1+x^2} \sup_{|t| \leq 2\sqrt[4]{1+x^2}} |t| \leq \frac{c}{\sqrt{1+x^2}}, \quad x \gg 1. \end{aligned} \quad (6.16)$$

The estimate for $\tau_2(x)$, $x \gg 1$, follows from (6.10) and (6.1):

$$\begin{aligned} |\tau_2(x)| &\leq \left| \int_0^{\sqrt{2}d(x)} q_2(x+t) dt \right| + \left| \int_0^{\sqrt{2}d(x)} q_2(x-t) dt \right| \\ &= \left| \int_{x-\sqrt{2}d(x)}^x q_2(\xi) d\xi \right| + \left| \int_x^{x+\sqrt{2}d(x)} q_2(\xi) d\xi \right| \\ &\leq 2 \sup_{[\alpha, \beta] \subseteq \omega(x)} \left| \int_\alpha^\beta \frac{\cos e^\xi}{1+\xi^2} d\xi \right| \leq c \frac{e^{-x}}{\sqrt{1+x^2}}, \quad x \gg 1. \end{aligned} \quad (6.17)$$

From (6.16), (6.17) and (6.13), we obtain (4.6), and therefore $q \in H$.

5. *Checking that the weights $\mu(x)$ and $\theta(x)$ agree with the function q .*

Equalities (4.12) for the functions $\mu(x)$ and $1/\theta(x)$ (see (6.2)) are easily proved with the help of estimates (6.13).

6. *Proof of assertion B).*

Below we use Theorem 4.7. Let us check that in case (6.2) requirements (4.2) are satisfied.

Let $x_0 \gg 1$. Then

$$\begin{aligned} \int_0^\infty \mu(t) dt &= \int_0^\infty \frac{dt}{\sqrt{1+t^2} \ln(2+t^2)} \geq \int_{x_0}^\infty \frac{1}{t\sqrt{1+t^{-2}}} \cdot \frac{dt}{2 \ln t + \ln(1+2t^{-2})} \\ &\geq c^{-1} \int_{x_0}^\infty \frac{dt}{t \ln t} = \infty \quad \Rightarrow \quad (4.2). \end{aligned}$$

Since the weights μ and θ agree with the function q , and one has the relations (see assertion (6.13)):

$$m(q, \mu, \theta) = \sup_{x \in \mathbb{R}} \left(\frac{\mu(x)}{\theta(x)} d^2(x) \right) \leq c \sup_{x \in \mathbb{R}} \left(\frac{\mu(x)}{\theta(x)} \sqrt{1+x^2} \right) = c < \infty,$$

assertion B) follows from Theorem 4.7.

□

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